On extremal graphs with at most ℓ internally disjoint Steiner trees connecting any $n-1$ vertices*

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Abstract

The concept of maximum local connectivity $\overline{\kappa}$ of a graph was introduced by Bollobás. One of the problems about it is to determine the largest number of edges $f(n;\overline{\kappa}\leq \ell)$ for graphs of order n that have local connectivity at most ℓ. We consider a generalization of the above concept and problem. For $S\subseteq V(G)$ and $|S|\geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G. The parameter $\overline{\kappa}_k(G)=max\{\kappa(S)|S\subseteq V(G),|S|=k\}$ is called the *maximum generalized local connectivity* of G. This paper it to consider the problem of determining the largest number $f(n;\overline{\kappa}_k\leq \ell)$ of edges for graphs of order n that have maximum generalized local connectivity at most ℓ. The exact value of $f(n;\overline{\kappa}_k\leq \ell)$ for $k=n,n-1$ is determined. For a general k, we construct a graph to obtain a sharp lower bound.

Keywords: (edge-)connectivity, Steiner tree, internally (edge-)disjoint trees, generalized local (edge-)connectivity.

AMS subject classification 2010: 05C40, 05C05, 05C35, 05C75.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to book [5] for graph theoretical notation and terminology not described here. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x,y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G)=\min\{\kappa_G(x,y)|x,y\in V(G),x\neq y\}$ is defined as *the connectivity* of G. In contrast to this parameter, $\overline{\kappa}(G)=\max\{\kappa_G(x,y)|x,y\in V(G),x\neq y\}$, introduced by Bollobás, is called the *maximum local connectivity* of G. The problem of determining the smallest number of edges, $h_1(n;\overline{\kappa}\geq r)$, which guarantees that any graph with n vertices and $h_1(n;\overline{\kappa}\geq r)$ edges will contain a pair of vertices joined by r internally disjoint paths was posed by Erdös and Gallai, see [1] for details. Bollobás [2] considered the problem of determining the largest number of edges, $f(n;\overline{\kappa}\leq \ell)$, for graphs with n vertices and local connectivity at most ℓ, that is, $f(n;\overline{\kappa}\leq \ell)=\max\{e(G)||V(G)|=n \ and \ \overline{\kappa}(G)\leq \ell\}$. One can see that $h_1(n;\overline{\kappa}\geq \ell+1)=f(n;\overline{\kappa}\leq \ell)+1$. Similarly, let $\lambda_G(x,y)$ denote the local edge-connectivity connecting x and y in G. Then $\lambda(G)=\min\{\lambda_G(x,y)|x,y\in V(G),x\neq y\}$ and $\overline{\lambda}(G)=\max\{\lambda_G(x,y)|x,y\in V(G),x\neq y\}$ are the edge-connectivity and maximum local edge-connectivity, respectively. So the edge version of the above problems can be given similarly. Set $g(n;\overline{\lambda}\leq \ell)=\max\{e(G)||V(G)|=n \ and \ \overline{\lambda}(G)\leq \ell\}$. Let

*Supported by NSFC No.11071130, and the “973” program.

$h_2(n;\overline{\lambda} \geq r)$ denote the smallest number of edges which guarantees that any graph with n vertices and $h_2(n;\overline{\kappa} \geq r)$ edges will contain a pair of vertices joined by r edge-disjoint paths. Similarly, $h_2(n;\overline{\lambda} \geq \ell+1) = g(n;\overline{\lambda} \leq \ell)+1$. The problem of determining the precise value of the parameters $f(n;\overline{\kappa} \leq \ell)$, $g(n;\overline{\lambda} \leq \ell)$, $h_1(n;\overline{\kappa} \geq r)$, or $h_2(n;\overline{\kappa} \geq r)$ has obtained wide attention and many results have been worked out; see [2, 3, 4, 8, 9, 10, 17, 18, 20].

In [11], we generalized the above classical problem. Before introducing our generalization, we need some basic concepts and notions. For a graph $G=(V,E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (*a Steiner tree* for short) is a such subgraph $T(V',E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are *internally disjoint* if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G. Note that when $|S|=2$ a Steiner tree connecting S is just a path connecting S. For an integer k with $2 \leq k \leq n$, the *generalized connectivity*, introduced by Chartrand et al. in 1984 [6], is defined as $\kappa_k(G) = min\{\kappa(S) | S \subseteq V(G), |S| = k\}$. For results on the generalized connectivity, see [12, 14, 13, 15]. Similar to the classical maximum local connectivity, we [11] introduced the parameter $\overline{\kappa}_k(G) = max\{\kappa(S) | S \subseteq V(G), |S| = k\}$, which is called the *maximum generalized local connectivity* of G. There we considered the problem of determining the largest number of edges, $f(n;\overline{\kappa}_k \leq \ell)$, for graphs with n vertices and maximal generalized local connectivity at most ℓ, that is, $f(n;\overline{\kappa}_k \leq \ell) = \max\{e(G) || V(G)| = n \ and\ \overline{\kappa}_k(G) \leq \ell\}$. We also considered the smallest number of edges, $h_1(n;\overline{\kappa}_k \geq r)$, which guarantees that any graph with n vertices and $h_1(n;\overline{\kappa}_k \geq r)$ edges will contain a set S of k vertices such that there are r internally disjoint S-trees. It is easy to check that $h_1(n;\overline{\kappa}_k \geq \ell+1) = f(n;\overline{\kappa}_k \leq \ell)+1$ for $0 \leq \ell \leq n - \lceil k/2 \rceil - 1$. In [11], we determine that $f(n;\overline{\kappa}_3 \leq 2) = 2n-3$ for $n \geq 3$ and $n \neq 4$, and $f(n;\overline{\kappa}_3 \leq 2) = 2n-2$ for $n=4$. Furthermore, we characterized graphs attaining these values. For general ℓ, we constructed graphs to show that $f(n;\overline{\kappa}_3 \leq \ell) \geq \frac{\ell+2}{2}(n-2)+\frac{1}{2}$ for both n and k odd, and $f(n;\overline{\kappa}_3 \leq \ell) \geq \frac{\ell+2}{2}(n-2)+1$ otherwise.

We continue to study the above problems in this paper. The edge version of these problems are also introduced and investigated. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized edge-connectivity* [16] is defined as $\lambda_k(G) = min\{\lambda(S) | S \subseteq V(G), |S| = k\}$. The parameter $\overline{\lambda}_k(G) = max\{\lambda(S) | S \subseteq V(G), |S| = k\}$ is called the *maximum generalized local edge-connectivity* of G. Similarly, $g(n;\overline{\lambda}_k \leq \ell) = \max\{e(G) || V(G)| = n \ and\ \overline{\lambda}_k(G) \leq \ell\}$, and $h_2(n;\overline{\lambda}_k \geq r)$ is the smallest number of edges, $h_2(n;\overline{\lambda}_k \geq r)$, which guarantees that any graph with n vertices and $h_2(n;\overline{\lambda}_k \geq r)$ edges will contain a set S of k vertices such that there are r edge-disjoint S-trees. Similarly, $h_2(n;\overline{\lambda}_k \geq \ell+1) = g(n;\overline{\lambda}_k \leq \ell)+1$ for $0 \leq \ell \leq n - \lceil k/2 \rceil - 1$.

The following result, due to Nash-Williams and Tutte, will be used later.

Theorem 1. *(Nash-Williams [19],Tutte [21]) A multigraph G contains a system of ℓ edge-disjoint spanning trees if and only if*

$$\|G/\mathscr{P}\| \geq \ell(|\mathscr{P}|-1)$$

holds for every partition $\mathscr{P}$ of $V(G)$, where $\|G/\mathscr{P}\|$ denotes the number of edges in G between distinct blocks of $\mathscr{P}$.

With the help of Theorem 1, this paper obtains the exact value of $f(n;\overline{\kappa}_k \leq \ell)$ and $g(n;\overline{\lambda}_k \leq \ell)$ for $k=n, n-1$. The graphs attaining these values are characterized. It is not easy to solve these problems for a general k ($3 \leq k \leq n$). So we construct a graph class to give them a sharp lower bound.

To start with, the following two observations are easily seen.

Observation 1. *Let G be a connected graph of order n. Then*

(1) $\kappa_k(G) \leq \lambda_k(G)$ *and* $\overline{\kappa}_k(G) \leq \overline{\lambda}_k(G)$*;*

(2) $\kappa_k(G) \leq \overline{\kappa}_k(G)$ *and* $\lambda_k(G) \leq \overline{\lambda}_k(G)$*.*

Observation 2. *If H is a spanning subgraph of G of order n, then $\kappa_k(H) \leq \kappa_k(G)$, $\lambda_k(H) \leq \lambda_k(G)$, $\overline{\kappa}_k(H) \leq \overline{\kappa}_k(G)$ and $\overline{\lambda}_k(H) \leq \overline{\lambda}_k(G)$.*

In [16], we obtained the exact value of $\lambda_k(K_n)$.

Lemma 1. *[16] Let n and k be two integers such that $3 \leq k \leq n$. Then*

$$\lambda_k(K_n) = n - \lceil k/2 \rceil$$

From Lemma 1, we can derive sharp bounds of $\overline{\lambda}_k(G)$.

Observation 3. *For a connected graph G of order n and $3 \leq k \leq n$, $1 \leq \overline{\lambda}_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

Proof. From the definitions of $\overline{\lambda}_k(G)$ and $\lambda_k(G)$ and the symmetricity of a complete graph, $\overline{\lambda}_k(K_n) = \lambda_k(K_n) = n - \lceil \frac{k}{2} \rceil$. So for a connected graph G of order n it follows that $\overline{\lambda}_k(G) \leq \overline{\lambda}_k(K_n) = n - \lceil \frac{k}{2} \rceil$. Since G is connected, $\overline{\lambda}_k(G) \geq 1$. So $1 \leq \overline{\lambda}_k(G) \leq n - \lceil \frac{k}{2} \rceil$. $\square$

One can easily check that the complete K_n attains the upper bound and any tree T of order n attains the lower bound. Combining Observation 3 with (1) of Observation 1, the following observation is immediate.

Observation 4. *For a connected graph G of order n and $3 \leq k \leq n$, $1 \leq \overline{\kappa}_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

2 The case $k = n$

In this section, we determine the exact value of $f(n; \overline{\lambda}_k \leq \ell)$ for the case $k = n$. This is also a preparation for the next section. From Observation 3, $1 \leq \overline{\lambda}_n(G) \leq \lfloor \frac{n}{2} \rfloor$. In order to make the parameter $f(n; \overline{\lambda}_k \leq \ell)$ to be meaningful, we assume that $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$. Let us focus on the the case $1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$ and begin with a lemma derived from Theorem 1.

Lemma 2. *Let G be a connected graph of order n ($n \geq 5$). If $e(G) \geq \binom{n-1}{2} + \ell$ ($1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor$) and $\delta(G) \geq \ell + 1$, then G contains $\ell + 1$ edge-disjoint spanning trees.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ ($1 \leq i \leq p$), and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in G. It suffices to show $|\mathcal{E}_p| \geq (\ell+1)(p-1)$ so that we can use Theorem 1.

The case $p = 1$ is trivial, thus we assume $p \geq 2$. For $p = 2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Then $|V_2| = n - n_1$. If $n_1 = 1$ or $n_1 = n-1$, then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \ell + 1$ since $\delta(G) \geq \ell + 1$. Suppose $2 \leq n_1 \leq n-2$. Then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \binom{n-1}{2} + \ell - \binom{n_1}{2} - \binom{n-n_1}{2} = -n_1^2 + nn_1 + \ell - (n-1)$. Since $2 \leq n_1 \leq n-2$, one can see that $|\mathcal{E}_2|$ attains its minimum value when $n_1 = 2$ or $n_1 = n-2$. Thus $|\mathcal{E}_2| \geq n - 3 + \ell \geq \ell + 1$. So the conclusion is true for $p = 2$ by Theorem 1.

Consider the case $p = n$. To have $|\mathcal{E}_n| \geq (\ell+1)(n-1)$, we must have $\binom{n-1}{2} + \ell \geq (\ell+1)(n-1)$, that is, $(n-2\ell-3)(n-2) \geq 2$. Since $\ell \leq \lfloor\frac{n-4}{2}\rfloor$, this inequality holds. The case $p = n-1$ can be proved similarly. Since $|\mathcal{E}_{n-1}| \geq \binom{n-1}{2} + \ell - 1$, we need the inequality $\frac{(n-1)(n-2)}{2} + \ell - 1 \geq (\ell+1)(n-2)$, that is, $(n-2\ell-3)(n-3) + (n-5) \geq 0$. Since $\ell \leq \lfloor\frac{n-4}{2}\rfloor$, this inequality holds.

Let us consider the remaining case p for $3 \leq p \leq n-2$. Clearly, $|\mathcal{E}_p| \geq e(G) - \sum_{i=1}^{p}\binom{n_i}{2} \geq \binom{n-1}{2} + \ell - \sum_{i=1}^{p}\binom{n_i}{2}$. We will show that $\binom{n-1}{2} + \ell - \sum_{i=1}^{p}\binom{n_i}{2} \geq (\ell+1)(p-1)$, that is, $\binom{n-1}{2} + \ell - (\ell+1)(p-1) \geq \sum_{i=1}^{p}\binom{n_i}{2}$. Actually, we only need to prove that $\frac{(n-1)(n-2)}{2} - (\ell+1)(p-2) - 1 \geq max\{\sum_{i=1}^{p}\binom{n_i}{2}\}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p}\binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n-p+1$, we need the inequality $\frac{(n-1)(n-2)}{2} - (\ell+1)(p-2) - 1 \geq \binom{1}{2}(p-1) + \binom{n-p+1}{2}$, that is, $(n-1)(n-2) - 2(\ell+1)(p-2) - 2 \geq (n-p+1)(n-p)$. Thus this inequality is equivalent to $(p-2)(2n-p-2\ell-3) \geq 2$. Since $1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor$ and $3 \leq p \leq n-2$, one can see that the inequality holds. Thus, $|\mathcal{E}_p| \geq (\ell+1)(p-1)$. From Theorem 1, we know that there exist $\ell+1$ edge-disjoint spanning trees, as desired. $\square$

In [16], the graphs with $\kappa_k(G) = n - \lceil\frac{k}{2}\rceil$ and $\lambda_k(G) = n - \lceil\frac{k}{2}\rceil$ were characterized, respectively.

Lemma 3. *[16] For a connected graph G of order n and $3 \leq k \leq n$, $\kappa_k(G) = n - \lceil\frac{k}{2}\rceil$ or $\lambda_k(G) = n - \lceil\frac{k}{2}\rceil$ if and only if $G = K_n$ for k even; $G = K_n \backslash M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

Note that $\kappa_n(G) = \lambda_n(G) = \overline{\kappa}_n(G) = \overline{\lambda}_n(G)$. From the above lemma, we can derive the following corollary.

Corollary 1. *For a connected graph G of order n, $\kappa_n(G) = \overline{\kappa}_n(G) = \lambda_n(G) = \overline{\lambda}_n(G) = \lfloor\frac{n}{2}\rfloor$ if and only if $G = K_n$ for n even; $G = K_n \setminus M$ for n odd, where M is an edge set such that $0 \leq |M| \leq \frac{n-1}{2}$.*

Let $\mathcal{G}_n$ be a graph class obtained from a complete graph K_{n-1} by adding a vertex v and joining v to ℓ vertices of K_{n-1}.

Theorem 2. *Let G be a connected graph of order n $(n \geq 6)$. If $\overline{\lambda}_n(G) \leq \ell$ $(1 \leq \ell \leq \lfloor\frac{n}{2}\rfloor)$, then*

$$e(G) \leq \begin{cases} \binom{n-1}{2} + \ell, & if\ 1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor; \\ \binom{n-1}{2} + n - 2, & if\ \ell = \lfloor\frac{n-2}{2}\rfloor \text{ and } n \text{ is even}; \\ \binom{n-1}{2} + \frac{n-3}{2}, & if\ \ell = \lfloor\frac{n-2}{2}\rfloor \text{ and } n \text{ is odd}; \\ \binom{n}{2}, & if\ \ell = \lfloor\frac{n}{2}\rfloor. \end{cases}$$

with equality if and only if $G \in \mathcal{G}_n$ for $1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor$; $G = K_n \setminus e$ where $e \in E(K_n)$ for $\ell = \lfloor\frac{n-2}{2}\rfloor$ and n even; $G = K_n \setminus M$ where $M \subseteq E(K_n)$ and $|M| = \frac{n+1}{2}$ for $\ell = \lfloor\frac{n-2}{2}\rfloor$ and n odd; $G = K_n$ for $\ell = \lfloor\frac{n}{2}\rfloor$.

Proof. For $1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor$, if $e(G) \geq \binom{n-1}{2} + (\ell+1)$, then $\delta(G) \geq \ell+1$. From Lemma 2, $\overline{\lambda}_n(G) \geq \ell+1$, which contradicts to $\overline{\lambda}_n(G) \leq \ell$. So $e(G) \leq \binom{n-1}{2} + \ell$ for $1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor$. For $\ell = \lfloor\frac{n-2}{2}\rfloor$ and n even, $e(G) \leq \binom{n-1}{2} + n - 2$ by Corollary 1. By the same reason, $e(G) \leq \binom{n-1}{2} + \frac{n-3}{2}$ for $\ell = \lfloor\frac{n-2}{2}\rfloor$ and n odd. If $\ell = \lfloor\frac{n}{2}\rfloor$, then for any connected graph G $\overline{\lambda}_k(G) \leq \ell$ by Observation 3. So $e(G) \leq \binom{n}{2}$.

Now we characterize the graphs attaining the upper bounds. Consider the case $1 \leq \ell \leq \lfloor\frac{n-4}{2}\rfloor$. Suppose that G is a connected graph such that $e(G) = \binom{n-1}{2} + \ell$. Clearly, $\delta(G) \geq \ell$. Assume $\delta(G) \geq \ell+1$. Since $e(G) = \binom{n-1}{2} + \ell$, G contains $\ell+1$ edge-disjoint spanning trees by Lemma 2, namely, $\overline{\lambda}_n(G) \geq \ell+1$, a contradiction. So $\delta(G) = \ell$, and hence there exists a vertex v such that $d_G(v) = \ell$. Clearly, $e(G-v) = \binom{n-1}{2}$. Thus $G - v$ is a clique of order $n-1$. Therefore, $G \in \mathcal{G}_n$. For n even and $\ell = \lfloor\frac{n-2}{2}\rfloor$, let $e(G) = \binom{n-1}{2} + n - 2$. Obviously, $G = K_n \setminus e$, where $e \in E(K_n)$. For n odd and $\ell = \lfloor\frac{n-2}{2}\rfloor$,

let $e(G) = \binom{n-1}{2} + \frac{n-3}{2}$. Clearly, $G = K_n \setminus M$, where $M \subseteq E(K_n)$ and $|M| = \frac{n+1}{2}$. For $\ell = \lfloor \frac{n}{2} \rfloor$, if $e(G) = \binom{n}{2}$, then $G = K_n$. $\square$

Corollary 2. *For $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$ and $n \geq 6$,*

$$f(n;\overline{\kappa}_n \leq \ell) = g(n;\overline{\lambda}_n \leq \ell) = \begin{cases} \binom{n-1}{2} + \ell, & \text{if } 1 \leq \ell \leq \lfloor \frac{n-4}{2} \rfloor \text{ or } \ell = \lfloor \frac{n-2}{2} \rfloor \text{ and } n \text{ is odd;} \\ \binom{n-1}{2} + 2\ell, & \text{if } \ell = \lfloor \frac{n-2}{2} \rfloor \text{ and } n \text{ is even;} \\ \binom{n}{2}, & \text{if } \ell = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

3 The case $k = n - 1$

Before giving our main results, we need some preparations. From Observation 4, we know that $1 \leq \overline{\kappa}_{n-1}(G) \leq \lfloor \frac{n+1}{2} \rfloor$. So we only need to consider $1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$. In order to determine the exact value of $f(n;\overline{\kappa}_{n-1} \leq \ell)$ for a general ℓ ($1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$), we first focus on the cases $\ell = \lfloor \frac{n+1}{2} \rfloor$ and $\lfloor \frac{n-1}{2} \rfloor$. This is also because by characterizing the graphs with $\overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ and $\lfloor \frac{n-1}{2} \rfloor$, we can deal with the difficult case $\ell = \lfloor \frac{n-3}{2} \rfloor$.

3.1 The subcases $\ell = \lfloor \frac{n+1}{2} \rfloor$ and $\ell = \lfloor \frac{n-1}{2} \rfloor$

Let us begin this subsection with a useful lemma in [16].

Let $S \subseteq V(G)$ such that $|S| = k$, and $\mathscr{T}$ be a maximum set of edge-disjoint trees in G connecting S. Let $\mathscr{T}_1$ be the set of trees in $\mathscr{T}$ whose edges belong to $E(G[S])$, and $\mathscr{T}_2$ be the set of trees containing at least one edge of $E_G[S, \bar{S}]$. Thus, $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ (Throughout this paper, $\mathscr{T}$, $\mathscr{T}_1$, $\mathscr{T}_2$ are defined in this way).

Lemma 4. *[16] Let $S \subseteq V(G)$, $|S| = k$ and T be a tree connecting S. If $T \in \mathscr{T}_1$, then T uses $k-1$ edges of $E(G[S]) \cup E_G[S, \bar{S}]$; If $T \in \mathscr{T}_2$, then T uses k edges of $E(G[S]) \cup E_G[S, \bar{S}]$.*

The following results can be derived from Lemma 4.

Lemma 5. *Let $G = K_n \setminus M$ be a connected graph of order n ($n \geq 4$), where $M \subseteq E(K_n)$.*

(1) *If n is odd and $|M| \geq 1$, then $\overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$;*

(2) *If n is even and $|M| \geq \frac{n}{2}$, then $\overline{\lambda}_{n-1}(G) < \frac{n}{2}$.*

Proof. (1) For any $S \subseteq V(G)$ such that $|S| = n-1$, obviously, $|\bar{S}| = 1$ and $e \in E(G[S]) \cup E_G[S, \bar{S}]$. Let $|\mathscr{T}_1| = x$ and $|\mathscr{T}| = y$. Then $|\mathscr{T}_2| = y - x$. Clearly, $|\mathscr{T}_1| \leq \lfloor \frac{\binom{n-1}{2}}{n-2} \rfloor = \frac{n-1}{2}$. Since $(n-2)|\mathscr{T}_1| + (n-1)|\mathscr{T}_2| \leq |E(G[S]) \cup E_G[S, \bar{S}]|$, it follows that $(n-2)x + (n-1)(y-x) \leq \binom{n}{2} - 1$. Then $\lambda(S) = |\mathscr{T}| = y \leq \frac{x}{n-1} + \frac{n}{2} - \frac{1}{n-1} \leq \frac{n+1}{2} - \frac{1}{n-1} < \frac{n+1}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$.

(2) In this case, for any $S \subseteq V(G)$ such that $|S| = n-1$, we have $|\bar{S}| = 1$ and $e \in E(G[S]) \cup E_G[S, \bar{S}]$. Let $|\mathscr{T}_1| = x$ and $|\mathscr{T}| = y$. Then $|\mathscr{T}_2| = y - x$. Clearly, $|\mathscr{T}_1| \leq \lfloor \frac{\binom{n-1}{2}}{n-2} \rfloor = \lfloor \frac{n-1}{2} \rfloor = \frac{n-2}{2}$. Since $(n-2)|\mathscr{T}_1| + (n-1)|\mathscr{T}_2| \leq |E(G[S]) \cup E_G[S, \bar{S}]|$, it follows that $(n-2)x + (n-1)(y-x) \leq \binom{n}{2} - \frac{n}{2}$. Then $\lambda(S) = |\mathscr{T}| = y \leq \frac{x}{n-1} + \frac{n}{2} - \frac{n}{2(n-1)} \leq \frac{n}{2} - \frac{1}{n-1} < \frac{n}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n}{2}$. $\square$

With the help of Lemmas 3 and 5 and Observation 1, the graphs with $\overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ can be characterized now.

Proposition 1. *For a connected graph G of order n ($n \geq 4$), $\overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ if and only if $G = K_n$ for n odd; $G = K_n \setminus M$ for n even, where M is an edge set such that $0 \leq |M| \leq \frac{n-2}{2}$.*

Proof. Consider the case n odd. Suppose that G is a connected graph such that $\overline{\kappa}_{n-1}(G) = \frac{n+1}{2}$. In fact, the complete graph K_n is a unique graph attaining this value. Let $G = K_n \setminus e$ where $e \in E(K_n)$. From (1) of Lemma 5 and Observation 1, $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$. Conversely, if $G = K_n$, then $\overline{\kappa}_{n-1}(G) \geq \kappa_{n-1}(G) = \frac{n+1}{2}$ by Lemma 3. Combining this with Observation 4, $\overline{\kappa}_{n-1}(G) = \frac{n+1}{2}$.

Now consider the case n even. Assume that G is a connected graph such that $\overline{\kappa}_{n-1}(G) = \frac{n}{2}$. If $G = K_n \setminus M$ such that $|M| \geq \frac{n}{2}$, then $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) < \frac{n}{2}$ by (2) of Lemma 5. So $0 \leq |M| \leq \frac{n-2}{2}$. Conversely, if $G = K_n \setminus M$ such that $0 \leq |M| \leq \frac{n-2}{2}$, then $\overline{\kappa}_{n-1}(G) \geq \kappa_{n-1}(G) = \frac{n}{2}$ by Lemma 3. From this together with Observation 4, $\overline{\kappa}_{n-1}(G) = \frac{n}{2}$. □

Proposition 2. *For a connected graph G of order n ($n \geq 4$), $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$ if and only if $G = K_n$ for n odd; $G = K_n \setminus M$ for n even, where M is an edge set such that $0 \leq |M| \leq \frac{n-2}{2}$.*

Proof. Assume that G is a connected graph satisfying the conditions of Proposition 2. From Observation 1 and Proposition 1, it follows that $\overline{\lambda}_{n-1}(G) \geq \overline{\kappa}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$. Combining this with Observation 3, $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$. Conversely, suppose $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n+1}{2} \rfloor$. For n odd, if $G = K_n \setminus e$ where $e \in E(K_n)$, then $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) < \frac{n+1}{2}$ by (1) of Lemma 5. So the complete graph K_n is a unique graph attaining this value. For n even, if $G = K_n \setminus M$ where $M \in E(K_n)$ such that $|M| \geq \frac{n}{2}$, then $\overline{\lambda}_{n-1}(G) < \lfloor \frac{n+1}{2} \rfloor$ by (2) of Lemma 5. So $0 \leq |M| \leq \frac{n-2}{2}$. □

We now focus our attention on the case $\ell = \lfloor \frac{n-1}{2} \rfloor$. Before characterizing the graphs with $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$, we need the following four lemmas. The notion of a second minimal degree vertex in a graph G will be used in the sequel. If G has two or more minimum degree vertices, then, choosing one of them as the first minimum degree vertex, a *second minimal degree vertex* is defined as any one of the rest minimum degree vertices of G. If G has only one minimum degree vertex, then a *second minimal degree vertex* is as its name, defined as any one of vertices that have the second minimal degree. Note that a second minimal degree vertex is usually not unique.

Lemma 6. *Let $G = K_n \setminus M$ be a connected graph of order n, where $M \subseteq E(K_n)$.*

(1) *If n ($n \geq 10$) is even and $|M| \geq \frac{3n-4}{2}$, then $\overline{\lambda}_{n-1}(G) < \frac{n-1}{2}$;*

(2) *If n ($n \geq 10$) is even, $n+1 \leq |M| \leq \frac{3n-6}{2}$ and there is a second minimal degree vertex, say u_1, such that $d_G(u_1) \leq \frac{n-4}{2}$, then $\overline{\lambda}_{n-1}(G) < \frac{n-2}{2}$;*

(3) *If n ($n \geq 8$) is odd and $|M| \geq n-1$, then $\overline{\lambda}_{n-1}(G) < \frac{n-1}{2}$.*

Proof. (1) For any $S \subseteq V(G)$ such that $|S| = n-1$, obviously, $|\bar{S}| = 1$ and $e \in E(G[S]) \cup E_G[S, \bar{S}]$. Set $S = V(G) \setminus v$ where $v \in V(G)$. Since G is connected graph, it follows that $d_G(v) \geq 1$ and hence $d_{K_n[M]}(v) \leq n-2$. So $|M \cap K_n[S]| \geq \frac{3n-4}{2} - (n-2) = \frac{n}{2}$ and $|E(G[S])| \leq \binom{n-1}{2} - \frac{n}{2}$. Therefore, $|\mathscr{T}_1| \leq \frac{\binom{n-1}{2} - \frac{n}{2}}{n-2} = \frac{n-2}{2} - \frac{1}{n-2} < \frac{n-2}{2}$, namely, $|\mathscr{T}_1| \leq \frac{n-4}{2}$. Let $|\mathscr{T}_1| = x$ and $|\mathscr{T}| = y$. Then $|\mathscr{T}_2| = y - x$. Since $(n-2)|\mathscr{T}_1| + (n-1)|\mathscr{T}_2| \leq |E(G[S]) \cup E_G[S, \bar{S}]|$, it follows that $(n-2)x + (n-1)(y-x) \leq \binom{n}{2} - \frac{3n-4}{2}$. Then $\lambda(S) = |\mathscr{T}| = y \leq \frac{x}{n-1} + \frac{n}{2} - \frac{3n-4}{2(n-1)} \leq \frac{n-2}{2} - \frac{1}{n-1} < \frac{n-2}{2}$. So $\overline{\lambda}_{n-1}(G) < \frac{n-2}{2}$.

(2) Let v be the vertex such that $d_G(v) = \delta(G)$. Then $d_G(v) \leq d_G(u_1) \leq \frac{n-4}{2}$. For any $S \subseteq V(G)$ with $|S| = n-1$, at least one of u_1, v belongs to S, say $u_1 \in S$. Hence $\overline{\lambda}_{n-1}(G) \leq \lambda(S) \leq d_G(u_1) \leq \frac{n-4}{2} < \frac{n-2}{2}$.

(3) The proof of (3) is similar to that of (1), and thus omitted. □

Lemma 7. *Let H be a connected graph of order $n-1$.*

(1) If n $(n \geq 5)$ is odd, $e(H) \geq \binom{n-2}{2}$, $\delta(H) \geq \frac{n-3}{2}$ and any two vertices of degree $\frac{n-3}{2}$ are nonadjacent, then H contains $\frac{n-3}{2}$ edge-disjoint spanning trees.

(2) If n $(n \geq 7)$ is even, $e(H) \geq \binom{n-2}{2} - \frac{n-2}{2}$, $\delta(H) \geq \frac{n-4}{2}$ and any two vertices of degree $\frac{n-4}{2}$ are nonadjacent, then H contains $\frac{n-4}{2}$ edge-disjoint spanning trees.

Proof. We only give the proof of (1), (2) can be proved similarly. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(H)$ with $|V_i| = n_i$ $(1 \leq i \leq p)$, and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in H. It suffices to show $|\mathcal{E}_p| \geq \frac{n-3}{2}(|\mathscr{P}|-1)$ so that we can use Theorem 1.

The case $p=1$ is trivial, thus we assume $p \geq 2$. For $p=2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Then $|V_2| = n-1-n_1$. If $n_1 = 1$ or $n_1 = n-2$, then $|\mathcal{E}_2| = |E_G[V_1,V_2]| \geq \frac{n-3}{2}$ since $\delta(H) \geq \frac{n-3}{2}$. Suppose $2 \leq n_1 \leq n-3$. Clearly, $|\mathcal{E}_2| = |E_G[V_1,V_2]| \geq \binom{n-2}{2} - \binom{n_1}{2} - \binom{n-1-n_1}{2} = -n_1^2 + (n-1)n_1 - (n-2)$. Since $2 \leq n_1 \leq n-3$, one can see that $|\mathcal{E}_2|$ attains its minimum value when $n_1 = 2$ or $n_1 = n-3$. Thus $|\mathcal{E}_2| \geq n-4 \geq \frac{n-3}{2}$ since $n \geq 5$. So the conclusion holds for $p=2$ by Theorem 1.

Now consider the remaining case p with $3 \leq p \leq n-1$. Since $|\mathcal{E}_p| \geq e(H) - \sum_{i=1}^{p}\binom{n_i}{2} \geq \binom{n-2}{2} - \sum_{i=1}^{p}\binom{n_i}{2}$, we need to show that $\binom{n-2}{2} - \sum_{i=1}^{p}\binom{n_i}{2} \geq \frac{n-3}{2}(p-1)$, that is, $\binom{n-2}{2} - \frac{n-3}{2}(p-1) \geq \sum_{i=1}^{p}\binom{n_i}{2}$. Furthermore, we only need to prove that $\binom{n-2}{2} - \frac{n-3}{2}(p-1) \geq max\{\sum_{i=1}^{p}\binom{n_i}{2}\}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p}\binom{n_i}{2}$ attains its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n-p$, we need the inequality $\binom{n-2}{2} - \frac{n-3}{2}(p-1) \geq \binom{1}{2}(p-1) + \binom{n-p}{2}$, that is, $(p-3)(n-p-1) \geq 0$. Since $3 \leq p \leq n-1$, one can see that the inequality holds. Thus, $|\mathcal{E}_p| \geq \frac{n-3}{2}(p-1)$. From Theorem 1, there exist $\frac{n-3}{2}$ edge-disjoint spanning trees. $\square$

The following theorem, due to Dirac, is well-known.

Theorem 3. *[5](p-485) Let G be a simple graph of order n $(n \geq 3)$ and minimum degree δ. If $\delta \geq \frac{n}{2}$, then G is Hamiltonian.*

Lemma 8. *If n $(n \geq 8)$ is odd and $G = K_n \setminus M$ such that $|M| = n-2$, then $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$.*

Proof. Clearly, $e(G) = \binom{n-1}{2} + 1$. Let v be the vertex such that $d_G(v) = \delta(G) = r$. Choose $S = V(G) \setminus v$. Then $|S| = n-1$. We distinguish the following cases to show this lemma.

Case 1. $1 \leq \delta(G) \leq \frac{n-1}{2}$.

If $\delta(G) = r = 1$, then $e(G-v) = \binom{n-1}{2}$, which implies that $G-v$ is a clique of order $n-1$. Obviously, $G[S]$ contains $\frac{n-1}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G-v) \geq \frac{n-1}{2}$. Therefore, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$.

Suppose $\delta(G) = r \geq 2$. Since $d_G(v) \leq \frac{n-1}{2}$, it follows that $d_{K_n[M]}(v) \geq n-1-\frac{n-1}{2} = \frac{n-1}{2}$. Combining this with $|M| = n-2$, $|M \cap E(K_n[S])| \leq n-2-\frac{n-1}{2} \leq \frac{n-3}{2}$, namely, $G[S]$ is a graph obtained from a clique of order $n-1$ by deleting at most $\frac{n-3}{2}$ edges. So $\delta(G[S]) \geq n-2-\frac{n-3}{2} = \frac{n-1}{2}$. Assume that there exists a vertex in S, say u_1, such that $d_{G[S]}(u_1) \leq \frac{n+1}{2}$. That is $d_{G[S]}(u_1) = \frac{n-1}{2}$ or $d_{G[S]}(u_1) = \frac{n+1}{2}$. Then $d_G(u_1) \leq \frac{n+3}{2}$, and hence $d_{K_n[M]}(u_1) \geq n-1-\frac{n+3}{2} = \frac{n-5}{2}$. We claim that the degree of each vertex of $S \setminus u_1$ is larger than $\frac{n+3}{2}$ in $G[S]$. Assume, to the contrary, that there exists a vertex in $S \setminus u_1$, say u_2, such that $d_{G[S]}(u_2) \leq \frac{n+1}{2}$. Then $d_G(u_2) \leq \frac{n+3}{2}$, and hence $d_{K_n[M]}(u_2) \geq \frac{n-5}{2}$. Therefore, $|M| \geq d_{K_n[M]}(v) + d_{K_n[M]}(u_1) + d_{K_n[M]}(u_2) \geq \frac{n-1}{2} + 2 \cdot \frac{n-5}{2} = \frac{3n-11}{2} > n-2$, a contradiction. From the above, we conclude that there exists at most one vertex in $G[S]$ such that its degree is $\frac{n-1}{2}$ or $\frac{n+1}{2}$. Since $\delta(G[S]) \geq \frac{n-1}{2}$, from Theorem 3 $G[S]$ is Hamiltonian and hence $G[S]$ contains a Hamilton cycle C. Let $S = \{u_1, u_2, \cdots, u_{n-1}\}$ such that $vu_i \in E(G)$ $(1 \leq i \leq r)$. Clearly, $vu_j \in M$ $(r+1 \leq j \leq n-1)$.

Then the vertices $u_1, u_2, \cdots, u_r$ divide the cycle C into r paths, say $P_1, P_2, \cdots, P_r$; see Figure 1 (a). We choose one edge $e_i \in E(P_i)$ $(1 \leq i \leq r)$ to delete that satisfies the following conditions:

❶ if there is no vertex of degree $\frac{n-1}{2}$ in $G[S]$, then e_i is chosen as any edge in P_i;

❷ if there exists one vertex u of degree $\frac{n-1}{2}$ in $G[S]$, then e_i is chosen as any edge in P_i that is incident with u.

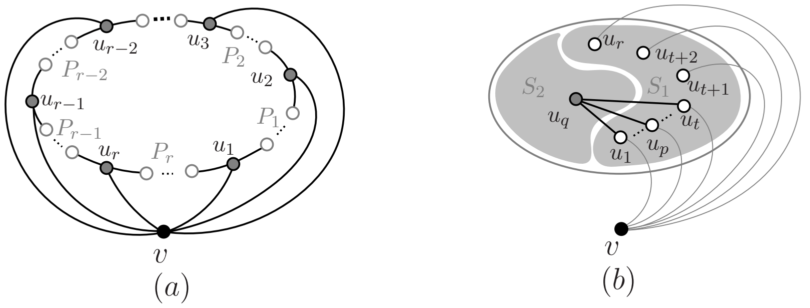


Figure 1. Graphs for Lemmas 8 and 9.

Then $T = vu_1 \cup vu_2 \cup \cdots vu_r \cup (P_1 \setminus e_1) \cup (P_2 \setminus e_2) \cdots (P_r \setminus e_r)$ is a Steiner tree connecting S. Set $G_1 = G \setminus E(T)$. Clearly, $\delta(G_1[S]) \geq \frac{n-3}{2}$ and there is at most one vertex of degree $\frac{n-3}{2}$. Combining this with $e(G_1[S]) = e(G) - (n-1) = \binom{n-1}{2} - (n-2) = \binom{n-2}{2}$, $G_1[S]$ contains $\frac{n-3}{2}$ spanning trees by (1) of Lemma 7. These trees together with the tree T are $\frac{n-1}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$.

Case 2. $\frac{n+1}{2} \leq \delta(G) \leq n-1$.

Let $S = V(G) \setminus v = \{u_1, \cdots, u_{n-1}\}$. Without loss of generality, let $S_1 = \{u_1, \cdots, u_r\}$ such that $vu_i \in E(G)$. Then $\frac{n+1}{2} \leq r \leq n-1$, and $S_2 = S \setminus S_1 = \{u_{r+1}, \cdots, u_{n-1}\}$. Since $d_G(v) = \delta(G) \geq \frac{n+1}{2}$, it follows that $|S_1| = r \geq \delta(G) \geq \frac{n+1}{2}$ and $|S_2| = n-1-r \leq n-1-\frac{n+1}{2} = \frac{n-3}{2}$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, u_j has at most $\frac{n-5}{2}$ neighbors in S_2 and hence $|E_G[u_j, S_1]| \geq \frac{n+1}{2} - \frac{n-5}{2} = 3$ since $d_G(u_j) \geq \delta(G) \geq \frac{n+1}{2}$. Clearly, the tree $T' = vu_1 \cup vu_2 \cup \cdots \cup vu_r$ is a Steiner tree connecting S_1. Our idea is to seek for $n-1-r$ edges in $E_G[S_1, S_2]$ and combine them with T' to form a Steiner tree connecting S. Choose the one with the smallest subscript among the maximum degree vertices in S_2, say u_1'. Then we search for the vertex adjacent to u_1' with the smallest subscript among all the maximum degree vertices in S_1, say u_1''. Let $e_1 = u_1'u_1''$. Consider the graph $G_1 = G \setminus e_1$. Pick up the one with the smallest subscript among all the maximum degree vertices in $S_2 \setminus u_1'$, say u_2'. Then we search for the vertex adjacent to u_2' with the smallest subscript among all the maximum degree vertices in S_1, say u_2''. Set $e_2 = u_2'u_2''$. We consider the graph $G_2 = G_1 \setminus e_2 = G \setminus \{e_1, e_2\}$. Choose the one with the smallest subscript among all the maximum degree vertices in $S_2 \setminus \{u_1', u_2'\}$, say u_3'. Then we search for the vertex adjacent to u_3' with the smallest subscript among all the maximum degree vertices in S_1, say u_3''. Let $e_3 = u_3'u_3''$. We now consider the graph $G_3 = G_2 \setminus e_3 = G \setminus \{e_1, e_2, e_3\}$. For each $u_i \in S_2$ $(r+1 \leq i \leq n-1)$, we proceed to find $e_4, e_5, \cdots, e_{n-1-r}$ in the same way. Let $M' = \{e_1, e_2, \cdots, e_{n-1-r}\}$ and $G_{n-1-r} = G \setminus M'$. Then $G_{n-1-r}[S] = G[S] \setminus M'$ and the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_r \cup e_1 \cup e_2 \cup \cdots \cup e_{n-1-r}$ is our desired tree. Set $G' = G \setminus E(T)$ (note that $G'[S] = G_{n-1-r}[S]$).

Claim 1. For each $u_j \in S_1$ $(1 \leq j \leq r)$, $d_{G'[S]}(u_j) \geq \frac{n-1}{2}$.

Proof of Claim 1. Assume, to the contrary, that there exists one vertex $u_p \in S_1$ such that $d_{G'[S]}(u_p) \leq \frac{n-3}{2}$. By the above procedure, there exists a vertex $u_q \in S_2$ such that when we pick up the edge $e_i = u_pu_q$ from $G_{i-1}[S]$ the degree of u_p in $G_i[S]$ is equal to $\frac{n-3}{2}$. That is $d_{G_i[S]}(u_p) = \frac{n-3}{2}$ and $d_{G_{i-1}[S]}(u_p) = \frac{n-1}{2}$. From our procedure, $|E_G[u_q, S_1]| = |E_{G_{i-1}}[u_q, S_1]|$. Without loss of generality, let $|E_G[u_q, S_1]| = t$ and

$u_qu_j \in E(G)$ for $1 \leq j \leq t$; see Figure 1 (b). Thus $u_p \in \{u_1, u_2, \cdots, u_t\}$. Recall that $|E_G[u_j, S_1]| \geq 3$ for each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$. Since $u_q \in S_2$, we have $t \geq 3$. Clearly, $u_qu_j \notin E(G)$ and hence $u_qu_j \in M$ for $t+1 \leq j \leq r$ by our procedure, namely, $|E_{K_n[M]}[u_q, S_1]| = r - t$. Since $d_{G_{i-1}[S]}(u_p) = \frac{n-1}{2}$, by our procedure $d_{G_{i-1}[S]}(u_j) \leq \frac{n-1}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Assume, to the contrary, that there is a vertex u_s $(1 \leq s \leq t)$ such that $d_{G_{i-1}[S]}(u_s) \geq \frac{n+1}{2}$. Then we should choose the edge u_qu_s instead of $e_i = u_qu_p$ by our procedure, a contradiction. We conclude that $d_{G_{i-1}[S]}(u_j) \leq \frac{n-1}{2}$ for each $u_j \in S_1$ $(1 \leq i \leq t)$. Clearly, there are at least $n - 2 - \frac{n-1}{2}$ edges incident to each u_j $(1 \leq j \leq t)$ that belong to $M \cup \{e_1, e_2, \cdots, e_{i-1}\}$. Since $i \leq n-1-r$, we have $\sum_{j=1}^{t} d_{K_n[M]}(u_j) \geq (n-2-\frac{n-1}{2})t - (i-1) > \frac{n-3}{2}t - (n-1-r)$ and hence $|M| \geq d_{K_n[M]}(v) + \sum_{j=1}^{t} d_{K_n[M]}(u_j) + |E_{K_n[M]}[u_q, S_1]| > (n-1-r) + \frac{n-3}{2}t - (n-1-r) + (r-t) = r + \frac{n-5}{2}t \geq \frac{n+1}{2} + \frac{3(n-5)}{2} = 2n - 7$, which contradicts to $|M| = n - 2$.

From Claim 1, $d_{G'[S]}(u_j) \geq \frac{n-1}{2}$ for each $u_j \in S_1$ $(1 \leq i \leq r)$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, $d_{G'[S]}(u_j) = d_{G[S]}(u_j) - 1 = d_G(u_j) - 1 \geq \delta(G) - 1 \geq \frac{n-1}{2}$. So $\delta(G'[S]) \geq \frac{n-1}{2}$. Combining this with $e(G'[S]) = e(G) - (n-1) = \binom{n-2}{2}$, $G'[S]$ contains $\frac{n-3}{2}$ spanning trees by From (1) of Lemma 7. These trees together with the tree T are $\frac{n-1}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-1}{2}$. □

Lemma 9. *If n $(n \geq 10)$ is even and $G = K_n \setminus M$ such that $|M| = \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$, then $\overline{\kappa}_{n-2}(G) \geq \frac{n-2}{2}$, where u_1 is a second minimal degree vertex in G.*

Proof. It is clear that $e(G) = \binom{n-2}{2} + \frac{n}{2} = \binom{n-1}{2} - \frac{n-4}{2}$. Let v be the vertex such that $d_G(v) = \delta(G) = r$. Let $S = V(G) \setminus v = \{u_1, \cdots, u_{n-1}\}$. Without loss of generality, let $S_1 = \{u_1, \cdots, u_r\}$ such that $vu_i \in E(G)$ $(1 \leq i \leq r)$. Then $S_2 = S \setminus S_1 = \{u_{r+1}, \cdots, u_{n-1}\}$ such that $vu_i \in M$ $(r+1 \leq i \leq n-1)$. We have the following two cases to consider.

Case 1. $1 \leq \delta(G) \leq \frac{n-2}{2}$.

If $d_G(v) = \delta(G) = 1$, then $e(G-v) = \binom{n-1}{2} - \frac{n-2}{2}$, which implies that $G - v$ is a graph obtained from a clique of order $n-1$ by deleting $\frac{n-2}{2}$ edges. From Corollary 1 and Observation 1, $\overline{\kappa}_{n-1}(G-v) = \frac{n-2}{2}$. Therefore, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$. Suppose $\delta(G) \geq 2$. Since $\delta(G) \leq \frac{n-2}{2}$, $d_{K_n[M]}(v) \geq n - 1 - \frac{n-2}{2} = \frac{n}{2}$ and hence $|M \cap K_n[S]| \leq n - 3$. Since $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex, we have $\delta(G[S]) \geq \frac{n-4}{2}$.

First, we consider the case $\delta(G[S]) \geq \frac{n}{2}$. We claim that there are at most two vertices of degree $\frac{n}{2}$ in $G[S]$. Assume, to the contrary, that there are three vertices of degree $\frac{n}{2}$ in $G[S]$, say u_1, u_2, u_3. Then $d_G(u_i) \leq \frac{n+2}{2}$ for $i = 1, 2, 3$ and hence $d_{K_n[M]}(u_i) \geq \frac{n-4}{2}$. Therefore, $|M| \geq d_{K_n[M]}(v) + \sum_{i=1}^{3} d_{K_n[M]}(u_i) \geq \frac{n}{2} + 3 \cdot \frac{n-4}{2} = \frac{4n-12}{2} = 2n - 6 > \frac{3n-6}{2}$, a contradiction. From the above, we conclude that there exist at most two vertices in $G[S]$ with degree $\frac{n}{2}$. Since $\delta(G[S]) \geq \frac{n}{2} > \frac{n-1}{2}$, from Theorem 3 $G[S]$ is Hamiltonian and hence $G[S]$ contains a Hamilton cycle C. Then the vertices $u_1, u_2, \cdots, u_r$ divide the cycle C into r paths, say $P_1, P_2, \cdots, P_r$. We choose one edge $e_i \in E(P_i)$ $(1 \leq i \leq r)$ to delete that satisfies the following conditions:

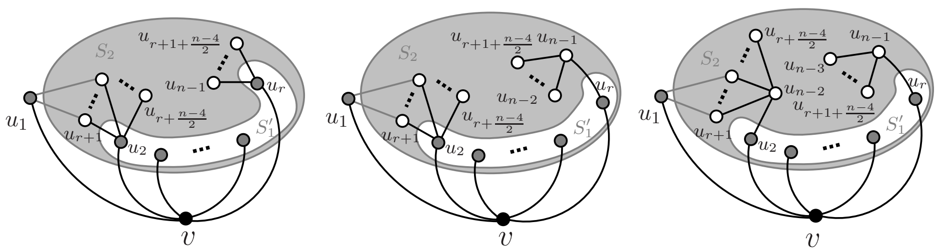
❶ if there are two vertices of degree $\frac{n}{2}$, say u_1, u_2 in $G[S]$, then e_i is chosen as any edge in P_i that is incident with at least one of u_1, u_2;

❷ if there is at most one vertex of degree $\frac{n}{2}$, then e_i is chosen as any edge in P_i.

Then $T = vu_1 \cup vu_2 \cup \cdots vu_r \cup (P_1 \setminus e_1) \cup (P_2 \setminus e_2) \cdots (P_r \setminus e_r)$ is a Steiner tree connecting S. Set $G_1 = G \setminus E(T)$. Obviously, $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$. Combining this with $e(G_1[S]) = e(G) - (n-1) = \binom{n-2}{2} - \frac{n-2}{2}$, $G_1[S]$ contains $\frac{n-4}{2}$ spanning trees by (2) of Lemma 7. These trees together with the tree T are $\frac{n-2}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$.

Next, we focus on the case that $\delta(G[S]) = \frac{n-2}{2}$ and $\delta(G[S]) = \frac{n-4}{2}$. If $\delta(G[S]) = \frac{n-4}{2}$, then there exists a vertex, say u_1, such that $d_{G[S]}(u_1) = \frac{n-4}{2}$. Since the degree of a second minimal degree vertex is not less than $\frac{n-2}{2}$, we have $u_1 \in S_1$. Thus $d_G(u_1) = \frac{n-2}{2}$ and $vu_1 \in E(G)$. If $\delta(G[S]) = \frac{n-2}{2}$, then there exists a vertex, say u_1, such that $d_{G[S]}(u_1) = \frac{n-2}{2}$ and $u_1 \in S_1$, or $d_{G[S]}(u_1) = \frac{n-2}{2}$ and $u_1 \in S_2$. Thus $d_G(u_1) = \frac{n}{2}$ and $u_1 \in S_1$, or $d_G(u_1) = \frac{n-2}{2}$ and $u_1 \in S_2$. We only give the proof of the case that $d_G(u_1) = \frac{n}{2}$ and $u_1 \in S_1$. The other two cases can be proved similarly.

Suppose $d_G(u_1) = \frac{n}{2}$ and $u_1 \in S_1$. Similar to the proof of Lemma 8, we want to find out a tree connecting S with root v, say T. Let $G_1 = G \backslash E(T)$. We hope that the graph $G_1[S]$ satisfies the conditions of (2) of Lemma 7. Thus there are $\frac{n-4}{2}$ spanning trees connecting S in $G_1[S]$. These trees together with the tree T are $\frac{n-2}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$. Let $S_1' = S_1 \backslash u_1$ and $S' = S_1' \cup S_2$. Let us focus on the graph $G[S_1']$. If $r = 2$, then $G[S']$ is a graph obtained from a clique of order $n-2$ by deleting one edge since $d_{K_n[M]}(u_1) = \frac{n-2}{2}$ and $d_{K_n[M]}(v) = n-3$ and $|M| = \frac{3n-6}{2}$. Without loss of generality, let $N_G(v) = \{u_1, u_2\}$. Clearly, $G[S']$ contains a Hamilton path P with u_2 as one of its endpoints. Then $T = vu_1 \cup vu_2 \cup P$. Set $G_1 = G \setminus E(T)$. Thus $\delta(G_1[S']) = \delta(G[S']) - 2 \geq n-4-2 = n-6 \geq \frac{n-2}{2}$. Combining this with $d_{G_1[S]}(u_1) = \frac{n-2}{2}$, the result follows by Lemma 7. Now assume $r \geq 3$. Since $d_{K_n[M]}(u_1) = \frac{n-2}{2}$, $d_{K_n[M]}(v) \geq \frac{n}{2}$ and $|M| = \frac{3n-6}{2}$, $G[S']$ is a graph obtained from the complete graph K_{n-2} by deleting at most $\frac{n-4}{2}$ edges and hence $\delta(G[S']) \geq n-3-\frac{n-4}{2} = \frac{n-2}{2}$. It is clear that there exist at least two vertices of degree $n-3$ and there is also at most one vertex of degree $\frac{n-2}{2}$ in $G[S']$. Without loss of generality, let u_{i_1}, u_{i_2} be two vertices of degree $n-3$.



(a) (b) (c)

Figure 2. Graphs for Case 1 of Lemma 9.

If $u_{i_1}, u_{i_2} \in S_1'$, without loss of generality, let $u_{i_1} = u_2$ and $u_{i_2} = u_r$, then the tree $T = vu_1 \cup \cdots \cup vu_r \cup u_2u_{r+1} \cup \cdots \cup u_2u_{r+\frac{n-4}{2}} \cup u_ru_{r+\frac{n-4}{2}+1} \cup \cdots \cup u_ru_{n-1}$ is a Steiner tree connecting S; see Figure 2 (a). Set $G_1 = G \setminus E(T)$. Clearly, $d_{G_1[S]}(u_1) = \frac{n-2}{2}$, $d_{G_1[S]}(u_2) \geq n-3-\frac{n-4}{2} \geq \frac{n-2}{2}$ and $d_{G_1[S]}(u_r) = (n-3) - (n-1-r-\frac{n-4}{2}) = r-2+\frac{n-4}{2} \geq \frac{n-2}{2}$. For $u_i \in S_2$ $(r+1 \leq i \leq n-1)$, $d_{G_1[S]}(u_i) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$. So $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$, as desired. If $u_{i_1} \in S_1'$ and $u_{i_2} \in S_2$, without loss of generality, let $u_{i_1} = u_2$ and $u_{i_2} = u_{n-1}$, then the tree $T = vu_1 \cup \cdots \cup vu_r \cup u_2u_{r+1} \cup \cdots \cup u_2u_{r+\frac{n-4}{2}} \cup u_{n-1}u_{r+\frac{n-4}{2}+1} \cup \cdots \cup u_{n-1}u_{n-2} \cup u_{n-1}u_r$ is our desired tree; see Figure 2 (b). Set $G_1 = G \setminus E(T)$. One can see that $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$, as desired. Let us consider the remaining case $u_{i_1}, u_{i_2} \in S_2$. Without loss of generality, let $u_{i_1} = u_{n-1}$ and $u_{i_2} = u_{n-2}$. The tree $T = vu_1 \cup \cdots \cup vu_r \cup u_{n-2}u_{r+1} \cup \cdots \cup u_{n-2}u_{r+\frac{n-4}{2}} \cup u_{n-1}u_{r+\frac{n-4}{2}+1} \cup \cdots \cup u_{n-1}u_{n-3} \cup u_2u_{n-2} \cup u_{n-1}u_r$ is our desired tree; see Figure 2 (c). Set $G_1 = G \setminus E(T)$. One can see that $\delta(G_1[S]) \geq \frac{n-4}{2}$ and there is at most one vertex of degree $\frac{n-4}{2}$ in $G_1[S]$. Using (2) of Lemma 7, we can get $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$.

Case 2. $\frac{n}{2} \leq \delta(G) \leq n-1$.

Recall that $S_1 = \{u_1, \cdots, u_r\}$ with $vu_i \in E(G)$ and $S_2 = S \setminus S_1 = \{u_{r+1}, \cdots, u_{n-1}\}$. Obviously, $|S_1| = r = \delta(G) \geq \frac{n}{2}$ and $|S_2| = n-1-r \leq n-1-\frac{n}{2} = \frac{n-2}{2}$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, u_j has

at most $\frac{n-4}{2}$ neighbors in S_2 and hence $|E_G[u_j, S_1]| \geq \frac{n}{2} - \frac{n-4}{2} = 2$ since $d_G(u_j) \geq \delta(G) \geq \frac{n}{2}$. Clearly, the tree $T' = vu_1 \cup vu_2 \cup \cdots \cup vu_r$ is a Steiner tree connecting S_1. Our idea is to seek for $n-1-r$ edges in $E_G[S_1, S_2]$ and combine them with T' to form a Steiner tree connecting S. We employ the method used in Case 2 of Lemma 8. Choose the one with the smallest subscript among all the maximum degree vertices in S_2, say u_1'. Then we search for the vertex adjacent to u_1' with the smallest subscript among all the maximum degree vertices in S_1, say u_1''. Let $e_1 = u_1'u_1''$. Consider the graph $G_1 = G \setminus e_1$. Pick up the one with the smallest subscript among all the maximum degree vertices in $S_2 \setminus u_1'$, say u_2'. Then we search for the vertex adjacent to u_2' with the smallest subscript among all the maximum degree vertices in S_1, say u_2''. Set $e_2 = u_2'u_2''$. We consider the graph $G_2 = G_1 \setminus e_1 = G \setminus \{e_1, e_2\}$. For each $u_i \in S_2$ $(r+1 \leq i \leq n-1)$, we proceed to find $e_3, e_4, \cdots, e_{n-1-r}$ in the same way. Let $M' = \{e_1, e_2, \cdots, e_{n-1-r}\}$ and $G_{n-1-r} = G \setminus M'$. Then $G_{n-1-r}[S] = G[S] \setminus M'$ and the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_r \cup e_1 \cup e_2 \cup \cdots \cup e_{n-1-r}$ is our desired tree. Set $G' = G \setminus E(T)$ (note that $G'[S] = G_{n-1-r}[S]$).

Claim 2. For each $u_j \in S_1$ $(1 \leq j \leq r)$, $d_{G'[S]}(u_j) \geq \frac{n-4}{2}$ and there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$.

Proof of Claim 2. First, we prove that for each $u_j \in S_1$ $(1 \leq j \leq r)$, $d_{G'[S]}(u_j) \geq \frac{n-4}{2}$. Assume, to the contrary, that there exists one vertex $u_p \in S_1$ such that $d_{G'[S]}(u_p) \leq \frac{n-6}{2}$. By the above procedure, there exists a vertex $u_q \in S_2$ such that when we pick up the edge $e_i = u_pu_q$ from $G_{i-1}[S]$ the degree of u_p in $G_i[S]$ is equal to $\frac{n-6}{2}$. That is $d_{G_i[S]}(u_p) = \frac{n-6}{2}$ and $d_{G_{i-1}[S]}(u_p) = \frac{n-4}{2}$. From our procedure, $|E_G[u_q, S_1]| = |E_{G_{i-1}}[u_q, S_1]|$. Without loss of generality, let $|E_G[u_q, S_1]| = t$ and $u_qu_j \in E(G)$ for $1 \leq j \leq t$; see Figure 1 (b). Thus $u_p \in \{u_1, u_2, \cdots, u_t\}$. Recall that $|E_G[u_j, S_1]| \geq 2$ for each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$. Since $u_q \in S_2$, we have $t \geq 2$. Clearly, $u_qu_j \notin E(G)$ and hence $u_qu_j \in M$ for $t+1 \leq j \leq r$ by our procedure, namely, $|E_{K_n[M]}[u_q, S_1]| = r-t$. Since $d_{G_{i-1}[S]}(u_p) = \frac{n-4}{2}$, by our procedure $d_{G_{i-1}[S]}(u_j) \leq \frac{n-4}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Assume, to the contrary, that there is a vertex u_s $(1 \leq s \leq t)$ such that $d_{G_{i-1}[S]}(u_s) \geq \frac{n-2}{2}$. Then we should choose the edge u_qu_s instead of $e_i = u_qu_p$ by our procedure, a contradiction. We conclude that $d_{G_{i-1}[S]}(u_j) \leq \frac{n-4}{2}$ for each $u_j \in S_1$ $(1 \leq i \leq t)$. Clearly, there are at least $n-2-\frac{n-4}{2}$ edges incident to each u_j $(1 \leq j \leq t)$ that belong to $M \cup \{e_1, e_2, \cdots, e_{i-1}\}$. Since $i \leq n-1-r$, we have $\sum_{j=1}^{p} d_{K_n[M]}(u_j) \geq (n-2-\frac{n-4}{2})t - (i-1) \geq \frac{n}{2}t - (n-2-r)$ and hence $|M| \geq d_{K_n[M]}(v) + \sum_{j=1}^{t} d_{K_n[M]}(u_j) + |E_{K_n[M]}[u_q, S_1]| \geq (n-1-r) + \frac{n}{2}t - (n-2-r) + (r-t) = r+1+\frac{n-2}{2}t \geq \frac{n}{2}+1+\frac{2(n-2)}{2} = \frac{3n-2}{2}$, which contradicts to $|M| = \frac{3n-6}{2}$.

Next, we consider to prove that there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$. Assume, to the contrary, that there exist two vertices of degree $\frac{n-4}{2}$ in $G'[S]$, say $u_{p'}, u_p$. By the above procedure, there exists a vertex $u_{q'} \in S_2$ such that when we pick up the edge $e_{i'} = u_{p'}u_{q'}$ from $G_{i'-1}[S]$ the degree of u_p in $G_{i'}[S]$ is equal to $\frac{n-4}{2}$, that is $d_{G_{i'}[S]}(u_{p'}) = \frac{n-4}{2}$. By the same reason, there exists a vertex $u_q \in S_2$ such that when we pick up the edge $e_i = u_pu_q$ from $G_{i-1}[S]$ the degree of u_p in $G_i[S]$ is equal to $\frac{n-4}{2}$, that is, $d_{G_i[S]}(u_p) = \frac{n-4}{2}$ and $d_{G_{i-1}[S]}(u_p) = \frac{n-2}{2}$. Without loss of generality, let $i' < i$. From our procedure, $|E_G[u_q, S_1]| = |E_{G_{i-1}}[u_q, S_1]|$. Without loss of generality, let $|E_G[u_q, S_1]| = t$ and $u_qu_j \in E(G)$ for $1 \leq j \leq t$; see Figure 1 (b). Thus $u_p \in \{u_1, u_2, \cdots, u_t\}$. Recall that $|E_G[u_j, S_1]| \geq 2$ for each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$. Since $u_q \in S_2$, we have $t \geq 2$. Then $u_qu_j \notin E(G)$ and hence $u_qu_j \in M$ for $t+1 \leq j \leq r$ by our procedure, namely, $|E_{K_n[M]}[u_q, S_1]| = r-t$. Since $d_{G_{i-1}[S]}(u_p) = \frac{n-2}{2}$, by our procedure $d_{G_{i-1}[S]}(u_j) \leq \frac{n-2}{2}$ for each $u_j \in S_1$ $(1 \leq j \leq t)$. Assume, to the contrary, that there is a vertex u_s $(1 \leq s \leq t)$ such that $d_{G_{i-1}[S]}(u_s) \geq \frac{n}{2}$. Then we should choose the edge u_qu_s instead of $e_i = u_qu_p$ by our procedure, a contradiction. We conclude that $d_{G_{i-1}[S]}(u_j) \leq \frac{n-2}{2}$ for each $u_j \in S_1$ $(1 \leq i \leq t)$. If $u_{p'} \in \{u_1, \cdots, u_t\}$, without loss of generality, let $u_{p'} = u_1$, then $d_{K_n[M]}(u_1) + \sum_{j=2}^{t} d_{K_n[M]}(u_j) \geq (n-2-d_{G_{i-1}[S]}(u_1)) + (n-2-\frac{n-2}{2})(t-1) - (i-1) \geq (n-2-d_{G_{i'}[S]}(u_1)) + \frac{n-2}{2}(t-1) - (i-1) \geq (n-2-\frac{n-4}{2}) + \frac{n-2}{2}(t-1) - (n-2-r) = \frac{n-2}{2}t - n + 3 + r$ since $i \leq n-1-r$. Since $t \geq 2$ and $r \geq \frac{n}{2}$, we have

$|M| \geq d_{K_n[M]}(v)+d_{K_n[M]}(u_1)+\sum_{j=2}^{t} d_{K_n[M]}(u_j)+|E_{K_n[M]}[u_q,S_1]| \geq (n-1-r)+(\frac{n-2}{2}t-n+3+r)+(r-t) = \frac{n-4}{2}t+r+2 \geq \frac{2(n-4)}{2}+\frac{n}{2}+2 \geq \frac{3n-4}{2}$, which contradicts to $|M| = \frac{3n-6}{2}$. If $u_{p'} \notin \{u_1, \cdots, u_t\}$, then $u_{p'} \in \{u_{t+1}, \cdots, u_r\}$ and $d_{K_n[M]}(u_{p'})+\sum_{j=1}^{t} d_{K_n[M]}(u_j) \geq (n-2-d_{G_{i-1}[S]}(u_{p'}))+(n-2-\frac{n-2}{2})t-(i-1) \geq (n-2-d_{G_{i'}[S]}(u_{p'}))+\frac{n-2}{2}t-(i-1) \geq (n-2-\frac{n-4}{2})+\frac{n-2}{2}t-(n-2-r) = \frac{n-2}{2}(t+1)-n+3+r$ since $i \leq n-1-r$. Since $t \geq 2$ and $r \geq \frac{n}{2}$, we have $|M| \geq d_{K_n[M]}(v)+d_{K_n[M]}(u_{p'})+\sum_{j=1}^{p} d_{K_n[M]}(u_j)+(|E_{K_n[M]}[u_q,S_1]|-1) \geq (n-1-r)+\frac{n-2}{2}(t+1)-n+3+r+(r-1-t) = r+1+\frac{n-4}{2}t+\frac{n-2}{2} \geq \frac{n}{2}+1+\frac{2(n-4)}{2}+\frac{n-2}{2} = 2n-4$, which contradicts to $|M| = \frac{3n-6}{2}$. The proof of this claim is complete.

From Claim 2, $d_{G'[S]}(u_j) \geq \frac{n-4}{2}$ for each $u_j \in S_1$ $(1 \leq i \leq r)$ and and there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$. For each $u_j \in S_2$ $(r+1 \leq j \leq n-1)$, $d_{G'[S]}(u_j) = d_{G[S]}(u_j)-1 = d_G(u_j)-1 \geq \delta(G)-1 \geq \frac{n-2}{2}$. So $\delta(G'[S]) \geq \frac{n-4}{2}$ and there exists at most one vertex of degree $\frac{n-4}{2}$ in $G'[S]$. Combining this with $e(G'[S]) = e(G)-(n-1) = \binom{n-2}{2}-\frac{n-2}{2}$, $G'[S]$ contains $\frac{n-4}{2}$ spanning trees by (2) of Lemma 7. These trees together with the tree T are $\frac{n-2}{2}$ trees connecting S, namely, $\overline{\kappa}_{n-1}(G) \geq \frac{n-2}{2}$. □

Proposition 3. *For a connected graph G of order n $(n \geq 11)$, $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$ if and only if $G = K_n \backslash M$ and $M \subseteq E(K_n)$ satisfies one of the following conditions:*

- $1 \leq |M| \leq n-2$ *for n odd;*
- $\frac{n}{2} \leq |M| \leq n$ *for n even;*
- $n+1 \leq |M| \leq \frac{3n-6}{2}$ *and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex in G for n even.*

Proof. For n odd, if G is a connected graph of order n such that $\overline{\kappa}_{n-1}(G) = \frac{n-1}{2}$, then we can consider G as the graph obtained from a complete graph K_n by deleting some edges. Set $G = K_n \setminus M$ where $M \subseteq E(K_n)$. From Proposition 1, $|M| \geq 1$. Combining this with (3) of Lemma 6, $1 \leq |M| \leq n-2$. For n even, if G is a connected graph of order n such that $\overline{\kappa}_{n-1}(G) = \frac{n-2}{2}$, then we let $G = K_n \setminus M$, where $M \subseteq E(K_n)$. From Proposition 1, $|M| \geq \frac{n}{2}$. Combining this with (1) of Lemma 6, $\frac{n}{2} \leq |M| \leq \frac{3n-6}{2}$. Furthermore, for $n+1 \leq |M| \leq \frac{3n-6}{2}$ we have $d_G(u_1) \geq \frac{n-2}{2}$ by (2) of Lemma 6, where u_1 is a second minimal degree vertex. So $\frac{n}{2} \leq |M| \leq n$, or $n+1 \leq |M| \leq \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$.

Conversely, assume that G is a graph satisfying one of the conditions of this proposition. Then we will show $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$. For n odd, $G = K_n \setminus M$ and $M \subseteq E(K_n)$ such that $1 \leq |M| \leq n-2$. In fact, we only need to show that $\overline{\kappa}_{n-1}(G) \geq \lfloor\frac{n-1}{2}\rfloor$ for $|M| = n-2$. It follows by Lemma 8. Combining with Proposition 1, $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$. For n even, $G = K_n \setminus M$ and $M \subseteq E(K_n)$ such that $\frac{n}{2} \leq |M| \leq n$, or $n+1 \leq |M| \leq \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex. Actually, for $\frac{n}{2} \leq |M| \leq n$, we claim that $d_G(u_1) \geq \frac{n-2}{2}$, where u_1 is a second minimal degree vertex. Otherwise, let $d_G(u_1) \leq \frac{n-4}{2}$. Let v be the vertex such that $d_G(v) = \delta(G)$. From the definition of the second minimal degree vertex, $d_G(v) \leq d_G(u_1) \leq \frac{n-4}{2}$ and hence $d_{K_n[M]}(v) \geq d_{K_n[M]}(u_1) \geq n-1-\frac{n-4}{2} = \frac{n+2}{2}$. Therefore, $|M| \geq d_{K_n[M]}(v)+d_{K_n[M]}(u_1) \geq n+2$, a contradiction. So we only need to show that $\overline{\kappa}_{n-1}(G) \geq \lfloor\frac{n-1}{2}\rfloor$ for $|M| = \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex. It follows by Lemma 9. From this together with Proposition 1, $\overline{\kappa}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$. □

Proposition 4. *For a connected graph G of order n $(n \geq 11)$, $\overline{\lambda}_{n-1}(G) = \lfloor\frac{n-1}{2}\rfloor$ if and only if $G = K_n \backslash M$ and $M \subseteq E(K_n)$ satisfies one of the following conditions.*

- $1 \leq |M| \leq n-2$ *for n odd;*
- $\frac{n}{2} \leq |M| \leq n$ *for n even;*
- $n+1 \leq |M| \leq \frac{3n-6}{2}$ *and $d_G(u_1) \geq \frac{n-2}{2}$ where u_1 is a second minimal degree vertex in G for n even.*

Proof. Assume that G is a connected graph satisfying the conditions of Proposition 4. From Observation 1 and Proposition 3, it follows that $\overline{\lambda}_{n-1}(G) \geq \overline{\kappa}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$. Combining this with Proposition 2, $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$. Conversely, if $\overline{\lambda}_{n-1}(G) = \lfloor \frac{n-1}{2} \rfloor$, then from Lemma 6 we have $G = K_n \setminus M$ for n odd, where M is an edge set such that $1 \leq |M| \leq n-2$; $G = K_n \setminus M$ for n even, where M is an edge set such that $\frac{n}{2} \leq |M| \leq n$, or $n+1 \leq |M| \leq \frac{3n-6}{2}$ and $d_G(u_1) \geq \frac{n-2}{2}$. □

3.2 The subcase $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$

Now we consider the case $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$.

Lemma 10. *Let H is a connected graph of order $n-1$ $(n \geq 12)$. If $e(H) = \binom{n-2}{2} + 2\ell - (n-1)$ $(1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor)$ and $\delta(H) \geq \ell$ and any two vertices of degree ℓ are nonadjacent, then H contains ℓ edge-disjoint spanning trees.*

Proof. Let $\mathscr{P} = \bigcup_{i=1}^{p} V_i$ be a partition of $V(G)$ with $|V_i| = n_i$ $(1 \leq i \leq p)$, and $\mathcal{E}_p$ be the set of edges between distinct blocks of $\mathscr{P}$ in G. It suffices to show $|\mathcal{E}_p| \geq \ell(|\mathscr{P}| - 1)$ so that we can use Theorem 1.

The case $p = 1$ is trivial, thus we assume $p \geq 2$. For $p = 2$, we have $\mathscr{P} = V_1 \cup V_2$. Set $|V_1| = n_1$. Then $|V_2| = n - 1 - n_1$. If $n_1 = 1$ or $n_1 = n-2$, then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \ell$ since $\delta(H) \geq \ell$. If $n_1 = 2$ or $n_1 = n-3$, then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \ell$ since $\delta(H) \geq \ell$ and any two vertices of degree ℓ are nonadjacent. Suppose $3 \leq n_1 \leq n-4$. Then $|\mathcal{E}_2| = |E_G[V_1, V_2]| \geq \binom{n-2}{2} + 2\ell - (n-1) - \binom{n_1}{2} - \binom{n-1-n_1}{2} = -n_1^2 + (n-1)n_1 + 2\ell - (2n-3)$. Since $3 \leq n_1 \leq n-4$, one can see that $|\mathcal{E}_2|$ attains its minimum value when $n_1 = 3$ or $n_1 = n-4$. Thus $|\mathcal{E}_2| \geq n - 9 + 2\ell \geq \ell$. So the conclusion holds for $p = 2$ by Theorem 1.

Consider the case $p = 3$. We will show $|\mathcal{E}_3| \geq 2\ell$. Let $\mathscr{P} = V_1 \cup V_2 \cup V_3$ and $|V_i| = n_i$ $(i = 1, 2, 3)$ where $n_1 + n_2 + n_3 = n - 1$. If there are two of n_1, n_2, n_3 that equals 1, say $n_1 = n_2 = 1$, then $|\mathcal{E}_3| \geq 2\ell$ since $\delta(H) \geq \ell$ and any two vertices of degree ℓ are nonadjacent. If there is at most one of n_1, n_2, n_3 that equals 1, then we need to prove that $|\mathcal{E}_3| \geq \binom{n-2}{2} + 2\ell - (n-1) - \sum_{i=1}^{3} \binom{n_i}{2} \geq 2\ell$. Since $f(n_1, n_2, n_3) = \sum_{i=1}^{3} \binom{n_i}{2}$ attains its maximum value when $n_1 = 1$, $n_2 = 2$ and $n_3 = n-4$, we need the inequality $\binom{n-2}{2} + 2\ell - (n-1) - \binom{n-4}{2} - 1 \geq 2\ell$. Since $n \geq 12$, the inequality holds. So the conclusion holds for $p = 3$ by Theorem 1. For $p = n-1$, we will show $|\mathcal{E}_{n-1}| \geq \ell(n-2)$ so that we can use Theorem 1. That is $\binom{n-2}{2} + 2\ell - (n-1) \geq \ell(n-2)$. Thus we need the inequality $(n-2-2\ell)(n-4) - n \geq 0$. Since $\ell \leq \lfloor \frac{n-5}{2} \rfloor$, the inequality holds. For $p = n-2$, we need to prove $|\mathcal{E}_{n-2}| \geq \ell(n-3)$. Clearly, $|\mathcal{E}_{n-2}| \geq \binom{n-2}{2} + 2\ell - (n-1) - 1 \geq \ell(n-3)$. Thus we need the inequality $(n-2-2\ell)(n-5) - 4 \geq 0$. Since $\ell \leq \lfloor \frac{n-5}{2} \rfloor$, this inequality holds.

Let us consider the remaining case p with $4 \leq p \leq n-4$. Clearly, we need to prove that $|\mathcal{E}_p| \geq \binom{n-2}{2} + 2\ell - (n-1) - \sum_{i=1}^{p} \binom{n_i}{2} \geq \ell(p-1)$, that is, $\frac{(n-2)(n-3)}{2} + 2\ell - (n-1) - \ell p + \ell \geq \sum_{i=1}^{p} \binom{n_i}{2}$. Since $f(n_1, n_2, \cdots, n_p) = \sum_{i=1}^{p} \binom{n_i}{2}$ achieves its maximum value when $n_1 = n_2 = \cdots = n_{p-1} = 1$ and $n_p = n - p$, we need the inequality $\frac{(n-2)(n-3)}{2} + 3\ell - (n-1) - \ell p \geq \frac{(n-p)(n-p-1)}{2}$. It is equivalent to $(2n - 2\ell - p - 4)(p-3) \geq 4$. One can see that the inequality holds since $\ell \leq \frac{n-5}{2}$ and $4 \leq p \leq n-4$. From Theorem 1, we know that there exist ℓ edge-disjoint spanning trees. □

Lemma 11. *Let G be a connected graph of order n $(n \geq 12)$. If $e(G) \geq \binom{n-2}{2} + 2\ell$ $(1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor)$, $\delta(G) \geq \ell + 1$ and any two vertices of degree $\ell + 1$ are nonadjacent, then $\overline{\kappa}_{n-1}(G) \geq \ell + 1$.*

Proof. The following claim can be easily proved.

Claim 3. $\Delta(G) \geq n - 4$.

Proof of Claim 3. Assume, to the contrary, that $\Delta(G) \leq n-5$. Then $(n-2)(n-3)+4\ell = 2e(G) \leq n\Delta(G) \leq n(n-5)$, which implies that $4\ell+6 \leq 0$, a contradiction.

From Claim 3, $n-4 \leq \Delta(G) \leq n-1$. Our basic idea is to find out a Steiner tree T connecting $S = V(G) \setminus v$, where $v \in V(G)$ such that $d_G(v) = \Delta(G)$. Let $G_1 = G \setminus E(T)$. Then we prove that $G_1[S]$ satisfies the conditions of Lemma 10 so that $G_1[S]$ contains ℓ spanning trees. These trees together with the tree T are $\ell+1$ internally disjoint trees connecting S, which implies that $\overline{\kappa}_{n-1}(G) \geq \ell+1$, as desired. We distinguish the following four cases to show this lemma.

If $\Delta(G) = n-1$, then there exists a vertex $v \in V(G)$ such that $d_G(v) = n-1$. Let $S = V(G) \setminus v = \{u_1, u_2, \cdots, u_{n-1}\}$. Then $T = u_1v \cup u_2v \cup \cdots \cup u_{n-1}v$ is a tree connecting S. Set $G_1 = G \setminus E(T)$. Since $\delta(G) \geq \ell+1$ and any two vertices of degree $\ell+1$ are nonadjacent, it follows that $\delta(G_1[S]) \geq \ell$ and any two vertices of degree ℓ are nonadjacent. From Lemma 10, $G_1[S]$ contains ℓ spanning trees, as desired.

Consider the case $\Delta(G) = n-4$. We claim that $\delta(G) \geq \ell+4$. Otherwise, let $\delta(G) \leq \ell+3$. Then there exists a vertex u such that $d_G(u) \leq \ell+3$. Then $2[\binom{n-2}{2}+2\ell] = 2e(G) = \sum_{u\in V(G)} d(u) \leq d_G(u)+(n-1)\Delta(G) \leq (\ell+3)+(n-1)(n-4)$, which results in $\ell \leq \frac{1}{3}$, a contradiction. Since $\Delta(G) = n-4$, there exists a vertex $v \in V(G)$ such that $d_G(v) = n-4$. Let $S = V(G) \setminus v = \{u_1, \cdots, u_{n-1}\}$ such that $vu_{n-1}, vu_{n-2}, vu_{n-3} \notin E(G)$. Pick up $u_i \in N_G(u_{n-1}), u_j \in N_G(u_{n-2}), u_k \in N_G(u_{n-3})$ (note that u_i, u_j, u_k are not necessarily different). Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-4} \cup u_iu_{n-1} \cup u_ju_{n-2} \cup u_ku_{n-1}$ is our desired. Set $G_1 = G \setminus E(T)$. Since $\delta(G) \geq \ell+4$, $G_1[S]$ contains at most one vertex of degree ℓ, as desired.

If $\Delta(G) = n-2$, then there exists a vertex of degree $n-2$ in G, say v. Let $S = G \setminus v = \{u_1, u_2, \cdots, u_{n-1}\}$ such that u_{n-1} is the unique vertex with $u_{n-1}v \notin E(G)$. Let $d_G(u_{n-1}) = x$. Without loss of generality, let $N_G(u_{n-1}) = \{u_1, \cdots, u_x\}$. Since $\delta(G) \geq \ell+1$, $x \geq \ell+1 \geq 2$. First, we consider the case $x \geq 3$. We claim that there exists a vertex, say u_i $(1 \leq i \leq x)$, such that $d_G(u_i) \geq \ell+3$. Otherwise, let $d_G(u_j) \leq \ell+2$ for each u_j $(1 \leq j \leq x)$. Then $(n-2)(n-3)+4\ell = 2e(G) \leq d_G(u_{n-1}) + d_G(v) + \sum_{j=1}^{x} d_G(u_j) + \sum_{j=x+1}^{n-2} d_G(u_j) \leq x+(n-2)+(\ell+2)x+(n-2-x)(n-2)$ and hence $x \leq \frac{2n-4\ell-4}{n-\ell-5}$. Since $x \geq 3$, $n+\ell-11 \leq 0$, which contradicts to $n \geq 12$. So there exists a vertex, say u_i $(1 \leq i \leq x)$, such that $d_G(u_i) \geq \ell+3$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_{n-1}u_i$ is our desired. Set $G_1 = G \setminus E(T)$. It is clear that $\delta(G_1[S]) \geq \ell$ and any two vertices of degree ℓ are nonadjacent, as desired. Next, we consider the case $x = 2$. Then $\ell = 1$, $d_G(u_{n-1}) = 2$ and $N_G(u_{n-1}) = \{u_1, u_2\}$. Let p be the number of vertices of degree 2 in G. We claim $0 \leq p \leq 3$. Otherwise, let $p \geq 4$. Then $2\binom{n-2}{2}+4 = 2e(G) = \sum_{v\in V(G)} d(v) \leq 2p+(n-p)(n-2)$ and hence $p \leq \frac{3n-10}{n-4}$. Since $p \geq 4$, it follows that $n \leq 6$, a contradiction. So $0 \leq p \leq 3$. If $p = 3$, then there are three vertices of degree 2, say v_1, v_2, v_3. Let $G_1 = G \setminus \{v_1, v_2, v_3\}$. Since the three vertices are pairwise nonadjacent, $|V(G_1)| = n-3$ and $e(G_1) = \binom{n-2}{2}+2-6 = \binom{n-2}{2}-4 > \binom{n-3}{2}$, a contradiction. So we can assume $0 \leq p \leq 2$. If $p = 2$, then there are two vertices of degree 2, say v_1, v_2. Let $G_1 = G \setminus \{v_1, v_2\}$. Then G_1 is a graph obtained from a clique of order $n-2$ by deleting 2 edges and hence $\overline{\kappa}_{n-2}(G_1) \geq \lfloor \frac{n-2}{2} \rfloor - 2 \geq 2$, that is, G_1 contains two spanning trees, say T_1', T_2'. Let $N_G(v_1) = \{u_1, u_2\}$, the trees $T_i = T_i' \cup v_1u_i$ $(i = 1, 2)$ are two Steiner trees connecting $S = V(G) \setminus v_2$, which implies that $\overline{\kappa}_{n-1}(G) \geq 2$. So we now assume $0 \leq p \leq 1$. Consider the case $p = 1$. If $d_G(u_{n-1}) = 2$, then $d_G(u_j) \geq 3$ for each u_j $(1 \leq j \leq n-2)$. Recall that $N_G(u_{n-1}) = \{u_1, u_2\}$, certainly we have $d_G(u_j) \geq 3$ $(j = 1, 2)$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_1u_{n-1}$ is a Steiner tree connecting $S = V(G) \setminus v$. Set $G_1 = G \setminus E(T)$. Clearly, $d_{G_1[S]}(u_1) \geq 1$, $d_{G_1[S]}(u_{n-1}) = 1$ and $u_1u_{n-1} \notin E(G_1[S])$. In addition, the degree of the other vertices in $G_1[S]$ is at least 2, as desired. Assume $d_G(u_{n-1}) \geq 3$. Let u_i be the vertex of degree 2 in $V(G) \setminus \{v, u_{n-1}\}$. If $u_i \in N_G(u_{n-1})$, then there is another vertex $u_j \in N_G(u_{n-1})$ such that $d_G(u_j) \geq 3$ since $p = 1$. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_ju_{n-1}$ is our desired. Set $G_1 = G \setminus E(T)$.

Obviously, $d_{G_1[S]}(u_i) = 1$, $d_{G_1[S]}(u_j) \geq 1$, $d_{G_1[S]}(u_{n-1}) \geq 2$, $u_iu_j \notin E(G_1[S])$ and the degree of the other vertices in $G_1[S]$ is at least 2, as desired. If $u_i \notin N_G(u_{n-1})$, then there exists a vertex $u_j \in N_G(u_{n-1})$ such that $d_G(u_j) \geq 3$ and $u_iu_j \notin E(G)$. Thus the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_ju_{n-1}$ is our desired. Set $G_1 = G \setminus E(T)$. Clearly, $d_{G_1[S]}(u_i) = 1$, $d_{G_1[S]}(u_t) \geq 1$, $d_{G_1[S]}(u_{n-1}) \geq 2$, $u_iu_j \notin E(G_1[S])$ and the degree of the other vertices in $G_1[S]$ is at least 2, as desired. For the remaining case $p = 0$, we choose a vertex $u_j \in N_G(u_{n-1})$ and the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-2} \cup u_ju_{n-1}$ is our desired. Set $G_1 = G \setminus E(T)$. Clearly, $\delta(G_1[S]) \geq 1$ and there is at most one vertex of degree 1, as desired.

Let us consider the remaining case $\Delta(G) = n-3$. Then there exists a vertex of degree $n-3$, say v. Let p be the number of vertices of degree $\ell+1$. Since $(n-2)(n-3)+4\ell = 2e(G) \leq p(\ell+1)+(n-p)(n-3)$, it follows that $p \leq \frac{2n-4\ell-6}{n-\ell-4}$. Consider the case $\ell \geq 2$. Since $p \leq \frac{2n-4\ell-6}{n-\ell-4}$, if $p \geq 2$ then $\ell \leq 1$, a contradiction. So $0 \leq p \leq 1$ for $2 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$. Let $V(G) \setminus v = \{u_1, \cdots, u_{n-1}\}$ such that $vu_{n-1}, vu_{n-2} \notin E(G)$. Without loss of generality, let $d_G(u_{n-1}) \geq d_G(u_{n-2})$. For vertex $u \in V(G)$, we choose $\ell+1$ vertices in $N_G(u)$, say $u_1, u_2, \cdots, u_{\ell+1}$ and the following claim can be easily proved.

Claim 4. For $\ell \geq 2$, there exists a vertex $u_i \in N_G(u)$ such that $d_G(u_i) \geq \ell + 4$ $(1 \leq i \leq \ell + 1)$.

Proof of Claim 4. Assume, to the contrary, that $d_G(u_j) \leq \ell + 3$ for each u_j $(1 \leq j \leq \ell + 1)$. Then $(n-2)(n-3) + 4\ell = 2e(G) \leq (\ell+1)(\ell+3) + (n-\ell-1)(n-3)$ and hence $(\ell-1)(n-3) \leq \ell^2 + 3$. So $n - 3 \leq \frac{\ell^2+3}{\ell-1} = \ell + 1 + \frac{4}{\ell-1} \leq \ell + 5 \leq \frac{n+5}{2}$, which contradicts to $n \geq 12$.

First, we consider the case $u_{n-1}u_{n-2} \in E(G)$. From the above, $0 \leq p \leq 1$ for $2 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$, that is, there is at most one vertex of degree $\ell + 1$ in G. If $d_G(u_{n-2}) = \ell + 1$, then $d_G(u_{n-1}) \geq \ell + 2$ and hence there exists a vertex $u_i \in N_G(u_{n-1}) \setminus u_{n-2}$ such that $d_G(u_i) \geq \ell + 4$ by Claim 4. Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-3} \cup u_iu_{n-1} \cup u_{n-1}u_{n-2}$ is a Steiner tree connecting $S = V(G) \setminus v$. Clearly, $d_{G_1[S]}(u_{n-1}) \geq d_G(u_{n-1}) - 2 \geq \ell$, $d_{G_1[S]}(u_{n-2}) = d_G(u_{n-2}) - 1 = \ell$ and $u_{n-2}u_{n-1} \notin E(G_1)$. In addition, $d_{G_1[S]}(u_i) \geq d_G(u_i) - 2 \geq \ell+2$ and $d_{G_1[S]}(u_j) \geq d_G(u_j) - 1 \geq \ell+1$ for each $u_j \in V(G) \setminus \{u_{n-1}, u_{n-2}, u_i, v\}$. Thus $\delta(G_1[S]) \geq \ell$ and any two vertices of degree ℓ are nonadjacent, as desired. If $d_G(u_{n-2}) \geq \ell + 2$, then $d_G(u_{n-1}) \geq d_G(u_{n-2}) \geq \ell + 2$. From Claim 4, there exist two vertices, say u_i, u_j, such that $u_i \in N_G(u_{n-1}) \setminus u_{n-2}$, $u_j \in N_G(u_{n-2}) \setminus u_{n-1}$, $d_G(u_i) \geq \ell + 4$ and $d_G(u_j) \geq \ell + 4$ (note that u_i, u_j are not necessarily different). Then the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-3} \cup u_iu_{n-1} \cup u_ju_{n-2}$ is our desired. Set $G_1 = G \setminus E(T)$. One can see that $G_1[S]$ satisfies the conditions of Lemma 10. So we can get $\overline{\kappa}_{n-1}(G) \geq \ell+1$. Next, we consider the case $u_{n-1}u_{n-2} \notin E(G)$. Then $d_G(u_{n-1}) \geq d_G(u_{n-2}) \geq \ell+1$. From Claim 4, there exist two vertices, say u_i, u_j, such that $u_i \in N_G(u_{n-1}) \setminus u_{n-2}$, $u_j \in N_G(u_{n-2}) \setminus u_{n-1}$, $d_G(u_i) \geq \ell + 4$ and $d_G(u_j) \geq \ell + 4$ (note that u_i, u_j are not necessarily different). Thus the tree $T = vu_1 \cup vu_2 \cup \cdots \cup vu_{n-3} \cup u_iu_{n-1} \cup u_ju_{n-2}$ is our desired. Set $G_1 = G \setminus E(T)$ and $S = V(G) \setminus v$. One can check that $\delta(G_1[S]) \geq \ell$ and there is at most one vertex of degree ℓ, as desired. Similar to the proof of the case $\Delta(G) = n - 2$, we can prove that the conclusion holds for $\ell = 1$. The proof is now complete. □

3.3 Results for the maximum generalized local (edge-)connectivity

Let $\mathcal{H}_n$ be a graph class obtained from the complete graph of order $n-2$ by adding two nonadjacent vertices and joining each of them to any ℓ vertices of K_{n-2}. The following theorem summarizes the results for a general ℓ.

Theorem 4. *Let G be a connected graph of order n ($n \geq 12$). If $\overline{\kappa}_{n-1}(G) \leq \ell$ ($1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor$), then*

$$e(G) \leq \begin{cases} \binom{n-2}{2} + 2\ell, & if\ 1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor; \\ \binom{n-2}{2} + n - 2, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor\ and\ n\ is\ odd; \\ \binom{n-2}{2} + n - 4, & if\ \ell = \lfloor \frac{n-3}{2} \rfloor\ and\ n\ is\ even; \\ \binom{n-1}{2} + n - 2, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor\ and\ n\ is\ odd; \\ \binom{n-1}{2} + \frac{n-2}{2}, & if\ \ell = \lfloor \frac{n-1}{2} \rfloor\ and\ n\ is\ even; \\ \binom{n}{2}, & if\ \ell = \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

with equality if and only if $G \in \mathcal{H}_n$ for $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$; $G = K_n \setminus M$ where $|M| = n - 1$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n odd; $G \in \mathcal{H}_n$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n even; $G = K_n \setminus e$ where $e \in E(K_n)$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n odd; $G = K_n \setminus M$ where $|M| = \frac{n}{2}$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n even; $G = K_n$ for $\ell = \lfloor \frac{n+1}{2} \rfloor$.

Proof. For $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$, we assume that $e(G) \geq \binom{n-2}{2} + 2\ell + 1$. Then the following claim is immediate.

Claim 5. $\delta(G) \geq \ell + 1$.

Proof of Claim 5. Assume, to the contrary, that $\delta(G) \leq \ell$. Then there exists a vertex $v \in V(G)$ such that $d_G(v) = \delta(G) \leq \ell$, which results in $e(G - v) \geq e(G) - \ell \geq \binom{n-2}{2} + \ell + 1$. Since $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$, it follows that $\overline{\kappa}_{n-1}(G - v) \geq \ell + 1$ by Theorem 2, which results in $\overline{\kappa}_{n-1}(G) \geq \ell + 1$, a contradiction.

From Claim 5, $\delta(G) \geq \ell+1$. If any two vertices of degree $\ell+1$ are nonadjacent, then $\overline{\kappa}_{n-1}(G) \geq \ell+1$ by Lemma 11, a contradiction. Assume that v_1 and v_2 are two vertices of degree $\ell+1$ such that $v_1v_2 \in E(G)$. Let $G_1 = G\setminus\{v_1, v_2\}$ and $V(G_1) = \{u_1, \cdots, u_{n-1}\}$. Then $e(G_1) \geq e(G)-(2\ell+1) = \binom{n-2}{2}$ and hence G_1 is a clique of order $n-2$. Then G_1 contains $\lfloor \frac{n-2}{2} \rfloor \geq \ell+1$ edge-disjoint spanning trees, say $T_1', T_2', \cdots, T_{\ell+1}'$. Without loss of generality, let $N_G(v_1) = \{u_1, u_2, \cdots, u_\ell, v_2\}$. Choose $S = \{u_1, u_2, \cdots, u_{n-1}, v_1\}$. Then $T_i = T_i' \cup v_1u_i$ ($1 \leq i \leq \ell$) together with $T_{\ell+1} = T_{\ell+1}' \cup v_1v_2 \cup v_2u_t$ are $\ell + 1$ internally disjoint trees connecting S where $u_t \in N_G(v_2) \setminus v_1$, which implies that $\overline{\kappa}_{n-1}(G) \geq \ell + 1$, a contradiction. So $e(G) \leq \binom{n-2}{2} + 2\ell$ for $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$. From Proposition 3, $e(G) \leq \binom{n-2}{2} + n - 2$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n odd, and $e(G) \leq \binom{n-2}{2} + n - 4$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n even. From Proposition 1, $e(G) \leq \binom{n-1}{2} + n - 2$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n odd, and $e(G) \leq \binom{n-1}{2} + \frac{n-2}{2}$ for $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n even. If $\ell = \lfloor \frac{n+1}{2} \rfloor$, then for any connected graph G it follows that $\overline{\kappa}_{n-1}(G) \leq \ell$ by Observation 4 and hence $e(G) \leq \binom{n}{2}$.

Now we characterize the graphs attaining these upper bounds. For $\ell = \lfloor \frac{n+1}{2} \rfloor$, if $e(G) = \binom{n}{2}$, then $G = K_n$. For $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n odd, if $e(G) = \binom{n-1}{2} + n - 2$, then $G = K_n \setminus e$ where $e \in E(K_n)$. For $\ell = \lfloor \frac{n-1}{2} \rfloor$ and n even, if $e(G) = \binom{n-1}{2} + \frac{n-2}{2}$, then $G = K_n \setminus M$ where $|M| = \frac{n}{2}$. For $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n odd, if $e(G) = \binom{n-2}{2} + n - 2$, then $G = K_n \setminus M$ where $|M| = n - 1$. Assume that $e(G) = \binom{n-2}{2} + n - 4$ for $\ell = \lfloor \frac{n-3}{2} \rfloor$ and n even. From Proposition 3, G is a graph obtained from the complete graph K_{n-2} by adding two nonadjacent vertices and adding $\frac{n-4}{2}$ edges between each of them and the complete graph K_{n-2}, that is, $G \in \mathcal{H}_n$.

Let us now focus on the case $1 \leq \ell \leq \lfloor \frac{n-5}{2} \rfloor$. Suppose $e(G) = \binom{n-2}{2}+2\ell$. Similar to Claim 5, $\delta(G) \geq \ell$. If $\delta(G) = \ell + 1$ and any two vertices of degree $\ell + 1$ are nonadjacent, then $\overline{\kappa}_{n-1}(G) \geq \ell + 1$ by Lemma 11, a contradiction. Let v_1 and v_2 be two vertices of degree $\ell + 1$ such that $v_1v_2 \in E(G)$. It is clear that $G_1 = G \setminus \{v_1, v_2\}$ is a graph obtained from the complete graph of order $n - 2$ by deleting an edge. For n odd, from Corollary 1 we have $\overline{\kappa}_{n-2}(G_1) = \lfloor \frac{n-2}{2} \rfloor = \frac{n-3}{2} \geq \ell + 1$ since $\ell \leq \lfloor \frac{n-5}{2} \rfloor = \frac{n-5}{2}$. For n even, from Corollary 1, it follows that $\overline{\kappa}_{n-2}(G_1) \geq \lfloor \frac{n-2}{2} \rfloor - 1 = \frac{n-4}{2} \geq \ell + 1$ since $\ell \leq \lfloor \frac{n-5}{2} \rfloor = \frac{n-6}{2}$. Clearly, G_1 contains $\ell + 1$ edge-disjoint spanning trees, say $T_1', T_2', \cdots, T_{\ell+1}'$. Set $N_G(v_1) = \{u_1, u_2, \cdots, u_\ell, v_2\}$. Then $T_i = T_i' \cup v_1u_i$ ($1 \leq i \leq \ell$) and $T_{\ell+1} = T_{\ell+1}' \cup v_1v_2 \cup v_1u_t$ are $\ell+1$ internally disjoint trees connecting $S = V(G) \setminus v_2$ where $u_t \in N_G(v_2) \setminus v_1$, which implies that $\overline{\kappa}_{n-1}(G) \geq \ell + 1$, a contradiction. Suppose

$\delta(G)=\ell$. If there exist two vertices of degree ℓ, say v_1, v_2, such that $v_1v_2 \in E(G)$. Set $G_1 = G\setminus\{v_1, v_2\}$. Then $|V(G_1)| = n-2$ and $e(G_1) = \binom{n-2}{2}+1$, a contradiction.

So we assume that any two vertices of degree ℓ are nonadjacent in G. Let p be the number of vertices of degree ℓ. The following claim can be easily proved.

Claim 6. $2 \leq p \leq 3$.

Proof of Claim 6. Assume $p \geq 4$. Then $2\binom{n-2}{2}+4\ell = 2e(G) = \sum_{v\in V(G)} d(v) \leq p\ell + (n-p)(n-1)$ and hence $p \leq \frac{4n-4\ell-6}{n-\ell-1}$. Since $p\geq 4$, it follows that $4n-4\ell-4 \leq 4n-4\ell-6$, a contradiction. Assume $p=1$, that is, G contains exact one vertex of degree ℓ, say v_1. Set $G_1 = G\setminus v_1$. Clearly, $e(G_1) = e(G)-\ell = \binom{n-2}{2}+\ell$. Since $\overline{\kappa}_{n-1}(G) \leq \ell$, it follows that $\overline{\kappa}_{n-1}(G_1) \leq \overline{\kappa}_{n-1}(G) \leq \ell$. From Theorem 2, G_1 is a graph obtained from a clique of order $n-2$ by adding a vertex of degree ℓ, say v_2. Since $p=1$ and $v_1v_2 \notin E(G)$, we have $d_G(v_1) = \ell+1$ and $d_G(v_2)=\ell$. Clearly, $G_1 = G\setminus\{v_1, v_2\}$ is a clique of order $n-2$. Thus G_1 contains $\lfloor\frac{n-2}{2}\rfloor \geq \ell+1$ edge-disjoint spanning trees, say $T_1', T_2', \cdots, T_{\ell+1}'$. Without loss of generality, let $N_G(v_1) = \{v_2, u_1, u_2, \cdots, u_\ell\}$. Then the trees $T_i = v_1u_i \cup T_i'$ $(1\leq i\leq \ell)$ together with $T_{\ell+1} = T_{\ell+1}' \cup v_1v_2 \cup v_2u_t$ form $\ell+1$ edge-disjoint trees connecting $S = V(G)\setminus v_2$, where $u_t \in N_G(v_2)\setminus v_1$. This implies that $\overline{\kappa}_{n-1}(G) \geq \ell+1$, a contradiction.

From Claim 6, we know that $p=2,3$. If $p=3$, then G contains three vertices of degree ℓ, say v_1, v_2, v_3. Set $G_1 = G\setminus\{v_1, v_2, v_3\}$. Then $|V(G_1)| = n-3$ and $e(G_1) = \binom{n-2}{2}+2\ell-3\ell = \binom{n-2}{2}-\ell > \binom{n-3}{2}$ since $1 \leq \ell \leq \lfloor\frac{n-5}{2}\rfloor$, a contradiction. If $p=2$, then G contains two vertices of degree ℓ, say v_1, v_2. Set $G_1 = G\setminus\{v_1, v_2\}$. Since v_1 and v_2 are nonadjacent, $e(G_1) = e(G)-2\ell = \binom{n-2}{2}$ and hence G_1 is a complete graph of order $n-2$, which implies that $G \in \mathcal{H}_n$. $\square$

The following corollary is immediate from Theorem 4.

Corollary 3. *For* $1 \leq \ell \leq \lfloor\frac{n+1}{2}\rfloor$ *and* $n \geq 12$,

$$f(n;\overline{\kappa}_{n-1}\leq \ell) = \begin{cases} \binom{n-2}{2}+2\ell, & if\ 1\leq \ell\leq \lfloor\frac{n-5}{2}\rfloor,\ or\ \ell=\lfloor\frac{n-3}{2}\rfloor \ and\ n\ is\ even;\\ \binom{n-2}{2}+2\ell+1, & if\ \ell=\lfloor\frac{n-3}{2}\rfloor\ and\ n\ is\ odd;\\ \binom{n-1}{2}+\ell, & if\ \ell=\lfloor\frac{n-1}{2}\rfloor\ and\ n\ is\ even;\\ \binom{n-1}{2}+2\ell-1, & if\ \ell=\lfloor\frac{n-1}{2}\rfloor\ and\ n\ is\ odd;\\ \binom{n}{2}, & if\ \ell=\lfloor\frac{n+1}{2}\rfloor. \end{cases}$$

Now we focus on the edge case.

Theorem 5. *Let G be a connected graph of order n $(n\geq 12)$. If $\overline{\lambda}_{n-1}(G)\leq \ell$ $(1\leq \ell\leq\lfloor\frac{n+1}{2}\rfloor)$, then*

$$e(G) \leq \begin{cases} \binom{n-2}{2}+2\ell, & if\ 1\leq\ell\leq\lfloor\frac{n-5}{2}\rfloor;\\ \binom{n-2}{2}+n-2, & if\ \ell=\lfloor\frac{n-3}{2}\rfloor\ and\ n\ is\ odd;\\ \binom{n-2}{2}+n-4, & if\ \ell=\lfloor\frac{n-3}{2}\rfloor\ and\ n\ is\ even;\\ \binom{n-1}{2}+n-2, & if\ \ell=\lfloor\frac{n-1}{2}\rfloor\ and\ n\ is\ odd;\\ \binom{n-1}{2}+\frac{n-2}{2}, & if\ \ell=\lfloor\frac{n-1}{2}\rfloor\ and\ n\ is\ even;\\ \binom{n}{2}, & if\ \ell=\lfloor\frac{n+1}{2}\rfloor. \end{cases}$$

with equality if and only if $G\in\mathcal{H}_n$ for $1\leq\ell\leq\lfloor\frac{n-5}{2}\rfloor$; $G=K_n\setminus M$ where $|M|=n-1$ for $\ell=\lfloor\frac{n-3}{2}\rfloor$ and n odd; $G\in\mathcal{H}_n$ for $\ell=\lfloor\frac{n-3}{2}\rfloor$ and n even; $G=K_n\setminus e$ where $e\in E(K_n)$ for $\ell=\lfloor\frac{n-1}{2}\rfloor$ and n odd; $G=K_n\setminus M$ where $|M|=\frac{n}{2}$ for $\ell=\lfloor\frac{n-1}{2}\rfloor$ and n even; $G=K_n$ for $\ell=\lfloor\frac{n+1}{2}\rfloor$.

Proof. Since $\overline{\lambda}_{n-1}(G)\leq\ell$ $(1\leq\ell\leq\lfloor\frac{n-5}{2}\rfloor)$, it follows that $\overline{\kappa}_{n-1}(G)\leq\overline{\lambda}_{n-1}(G)\leq\ell$ and hence $e(G)\leq\binom{n-2}{2}+2\ell$ by Theorem 4. Suppose $e(G)=\binom{n-2}{2}+2\ell$. Since $\overline{\kappa}_{n-1}(G)\leq\overline{\lambda}_{n-1}(G)\leq\ell$, we have $G\in\mathcal{H}_n$

by Theorem 4. For $\ell = \lfloor\frac{n+1}{2}\rfloor$, $\lfloor\frac{n-1}{2}\rfloor$ and $\lfloor\frac{n-3}{2}\rfloor$, respectively, the conclusion holds by Propositions 2 and 4.

$\square$

Corollary 4. *For $1 \leq \ell \leq \lfloor\frac{n+1}{2}\rfloor$ and $n \geq 12$,*

$$g(n; \overline{\lambda}_{n-1} \leq \ell) = \begin{cases} \binom{n-2}{2} + 2\ell, & if\ 1 \leq \ell \leq \lfloor\frac{n-5}{2}\rfloor,\ or\ \ell = \lfloor\frac{n-3}{2}\rfloor \text{ and } n \text{ is even;} \\ \binom{n-2}{2} + 2\ell + 1, & if\ \ell = \lfloor\frac{n-3}{2}\rfloor \text{ and } n \text{ is odd;} \\ \binom{n-1}{2} + \ell, & if\ \ell = \lfloor\frac{n-1}{2}\rfloor \text{ and } n \text{ is even;} \\ \binom{n-1}{2} + 2\ell - 1, & if\ \ell = \lfloor\frac{n-1}{2}\rfloor \text{ and } n \text{ is odd;} \\ \binom{n}{2}, & if\ \ell = \lfloor\frac{n+1}{2}\rfloor. \end{cases}$$

Remark. It is not easy to determine the exact value of $f(n; \overline{\kappa}_k \leq \ell)$ and $g(n; \overline{\lambda}_k \leq \ell)$ for a general k. So we hope to give a sharp lower bound of them. We construct a graph G of order n as follows: Choose a complete graph K_{k-1} ($1 \leq \ell \leq \lfloor\frac{k-1}{2}\rfloor$). For the remaining $n-k+1$ vertices, we join each of them to any ℓ vertices of K_{k-1}. Clearly, $\overline{\kappa}_{n-1}(G) \leq \overline{\lambda}_{n-1}(G) \leq \ell$ and $e(G) = \binom{k-1}{2} + (n-k+1)\ell$. So $f(n; \overline{\kappa}_k \leq \ell) \geq \binom{k-1}{2} + (n-k+1)\ell$ and $g(n; \overline{\lambda}_k \leq \ell) \geq \binom{k-1}{2} + (n-k+1)\ell$. From Theorems 4 and 5, we know that these two bounds are sharp for $k = n, n-1$.

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